Research Statement

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1. **Introduction**

My main field of interest is *nonlinear partial differential equations* (PDEs) with tools from the modulation theory.

More specifically, I work about blow-up dynamics for parabolic problems, in particular in the *critical* setting. The first part of my thesis was the study of the blow-up dynamics for the 1-corotational energy critical harmonic heat flow [22], [24]. Next, in the continuation of this study and [12], I have worked on type II blow-up for the four fimensional energy critical semi linear heat equation [25]. In the second part, I have studied the two dimensional parabolic-elliptic Patlak-Keller-Segel model of chemotactic aggregation for radially symmetric initial data [23]. Currently, I am working about the parabolic-parabolic model in order to obtain a blow-up dynamic in finite time for radial initial data. I present in the sequel the main results with some natural perspectives in the continuation of these works.

2. **Harmonic heat flow**

2.1. **Introduction.** The harmonic heat flow between two embedded Riemanian manifolds $(N, g_N), (M, g_M)$ is the gradient flow associated to the Dirichlet energy of maps from $N \to M$:

$$\begin{cases} \partial_t v = \mathbb{P}_{T_v M}(\Delta_{g_N} v) \\ v_{|t=0} = v_0 \end{cases} \quad (t,x) \in \mathbb{R} \times N, \quad v(t,x) \in M \tag{2.1}$$

where $\mathbb{P}_{T_v M}$ is the projection onto the tangent space to M at v. The special case $N = \mathbb{R}^2$, $M = \mathbb{S}^2$ corresponds to the harmonic heat flow to the 2-sphere

$$\partial_t v = \Delta v + |\nabla v|^2 v, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2, \quad v(t,x) \in \mathbb{S}^2. \tag{2.2}$$

and is related to the Landau Lifschitz equation of ferromagnetism. We restrict our discussion to this case in order to clarify the results. Note that the Dirichlet energy is dissipated by the flow

$$\frac{d}{dt}\left\{ \int_{\mathbb{R}^2} |\nabla v|^2 \right\} = -2 \int_{\mathbb{R}^2} |\partial_t v|^2$$

and left invariant by the scaling symmetry

$$u_\lambda(t,x) = u(\lambda^2 t, \lambda x).$$

Hence the problem is *energy critical.*

2.2. ***k*-corotational solution.** Given a homotopy degree $k \in \mathbb{Z}^*$, the k-corotational solutions of (2.2) corresponds to solutions of the form:

$$v(t,r) = \left| \begin{array}{l} g(u(t,r)) \cos(k\theta) \\ g(u(t,r)) \sin(k\theta) \\ z(u(t,r)) \end{array} \right. \tag{2.3}$$

with u is solution to the following semilinear parabolic equation:

$$\begin{cases} \partial_t u - \partial_r^2 u - \frac{\partial_r u}{r} + k^2 \frac{\sin 2u}{2r^2} = 0, \\ u_{t=0} = u_0. \end{cases} \tag{2.4}$$

The k- corotational symmetry is preserved by the flow. Moreover the k-corotational Dirichlet energy becomes

$$E(u) = \int_0^{+\infty} \left[|\partial_r u|^2 + k^2 \frac{(\sin u)^2}{r^2} \right] r dr \tag{2.5}$$

and is minimized along maps with boundary conditions

$$u(0) = 0, \quad \lim_{r \to +\infty} u(r) = \pi \tag{2.6}$$

onto the least harmonic map Q which is the unique -up to scaling- solution to

$$r \partial_r Q_k = k g(Q_k) \tag{2.7}$$

satisfying (2.6), see for example [6]. In the special case of $\mathbb{S}^2$ target, the harmonic map is explicitly given by

$$Q_k(r) = 2 \tan^{-1}(r^k). \tag{2.8}$$

Let $\mathcal{Q}_k$ the least energy k-corotational harmonic map generated by the Q_k solution to (2.7), explicitely:

$$\mathcal{Q}_k = \left| \begin{array}{l} \frac{2r^k}{1+r^{2k}} cos(k\theta) \\ \frac{2r^k}{1+r^{2k}} sin(k\theta). \\ \frac{1-r^{2k}}{1+r^{2k}} \end{array} \right. \tag{2.9}$$

For corotational data and homotopy number $k \geq 2$, Guan, Gustaffson, Tsai [9], Gustaffson, Nakanishi, Tsai [10] prove that the flow is globally defined near the ground state harmonic map. In fact, $\mathcal{Q}_k$ is asymptotically stable for $k \geq 3$, and in particular no blow up will occur, and eternally oscillating solutions and infinite time grow up solutions are exhibited for $k = 2$. For the 1-corotational solutions, Van den Berg, Hulshof and King [1] implement a formal analysis based on the matched asymptotics techniques and predict the existence of blow up solutions of the form

$$u(t, r) \sim Q\left(\frac{r}{\lambda(t)}\right) \tag{2.10}$$

with blow up speed governed by the quantized rates

$$\lambda(t) \sim \frac{(T-t)^L}{|\log(T-t)|^{\frac{2L}{2L-1}}}, \quad L \in \mathbb{N}^*.$$

2.3. **Results and perspectives.** In [22], we prove the following theorem:

Theorem 2.1 (Stable blow up dynamics for the 1-corotational heat flow)**.** *Let $k = 1$. Let $\mathcal{Q}_1$ be the least energy harmonic map given by* (2.9)*. Then there exists an open set $\mathcal{O}$ of 1-corotational initial data of the form*

$$v_0 = \mathcal{Q}_1 + \varepsilon_0, \quad \varepsilon_0 \in \mathcal{O} \subset \dot{H}^1 \cap \dot{H}^4$$

such that the corresponding solution $v \in \mathcal{C}([0, T), \dot{H}^1 \cap \dot{H}^4)$ to (2.1) *blows up in finite time $0 < T = T(u_0) < +\infty$ according to the following universal scenario:*
(i) Universality of the concentrating bubble*: there exists an asympotic profile $v^* \in \dot{H}^1$ and $\lambda \in \mathcal{C}^1([0, T), \mathbb{R}_+^*)$ such that*

$$\lim_{t \to T} \left\| v(t, x) - \mathcal{Q}_1\left(\frac{x}{\lambda(t)}\right) - v^* \right\|_{\dot{H}^1} = 0. \tag{2.11}$$

(ii) Sharp asymptotics*: the blow up speed is given by*

$$\lambda(t) = c(v_0)(1+o(1))\frac{T-t}{|\log(T-t)|^2} \quad \text{as} \quad t \to T \tag{2.12}$$

for some $c(v_0) > 0$.
(iii) Regularity of the asymptotic profile*: there holds the additional regularity*

$$v^* \in \dot{H}^2. \tag{2.13}$$

This result lies in the continuation of the works [21], [18] on the derivation of stable or codimension one blow up dynamics for the wave map:

$$\text{(WM)} \quad \begin{cases} \partial_{tt}u - \Delta u = (|\partial_t u|^2 - |\nabla u|^2)u \\ u_{|t=0} = u_0, \quad \partial_t u_{|t=0} = u_1, \end{cases} \quad (t,x) \in \mathbb{R}\times\mathbb{R}^2, \quad u(t,x) \in \mathbb{S}^2, \tag{2.14}$$

and the Schrödinger map:

$$\text{(SM)} \quad \begin{cases} u \wedge \partial_t u = \Delta u + |\nabla u|^2 u \\ u_{|t=0} = u_0, \end{cases} \quad (t,x) \in \mathbb{R}\times\mathbb{R}^2, \quad u(t,x) \in \mathbb{S}^2, \tag{2.15}$$

The proof relies on an explicit construction of the focusing bubble which turns out to be a very robust approach. More recently, we have completed this result with the following theorem [24] :

Theorem 2.2 (Excited slow blow up dynamics for the 1-corotational heat flow)**.** *Let* $k = 1$*. Let* $\mathcal{Q}_1$ *be the least energy harmonic map. Let* $L \in \mathbb{N}^*$*. Then there exists a smooth corotational initial data* $u_0(r)$ *such that the corresponding solution to* (2.4) *blows up in finite time* $T = T(u_0) > 0$ *by bubbling off a harmonic map:*

$$\nabla u(t,r) - \nabla \mathcal{Q}_1\left(\frac{r}{\lambda(t)}\right) \to \nabla u^* \quad \text{in} \quad L^2 \quad \text{as} \quad t \to T \tag{2.16}$$

at the excited rate:

$$\lambda(t) = c(u_0)(1+o_{t\to T}(1))\frac{(T-t)^L}{|\log(T-t)|^{\frac{2L}{2L-1}}}, \quad c(u_0) > 0. \tag{2.17}$$

Moreover, u_0 *can be taken arbitrarily close to* $\mathcal{Q}_1$ *in the energy critical topology.*

We identify two main directions of research:

- There are probably other dynamics around the ground state. A challenging program is to classify the flow near the harmonic map, at least in corotational symmetry.
- Extension to the super critical dimension $N \geq 3$.

3. **Semilinear heat equation**

3.1. **Introduction.** In this section, we study the following semilinear heat equation

$$\partial_t u = \Delta u + u^p, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N, \tag{3.1}$$

which admits a dissipated energy

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}. \tag{3.2}$$

Moreover, we have a scaling invariance: if $u(t,x)$ solves (3.1), then so does $u_\lambda(t,x) = \lambda^{\frac{2}{p-1}}u(\lambda^2 t, \lambda x)$, with $\lambda > 0$. The critical space is a fundamental phenomenological number for the analysis and is defined as the number of derivatives in L^2 which are left invariant by the scaling symmetry of the flow:

$$\|u_\lambda(t)\|_{\dot{H}^{s_c}} = \|u(\lambda^2 t)\|_{\dot{H}^{s_c}}. \tag{3.3}$$

Hence

$$s_c = \frac{N}{2} - \frac{2}{p-1}. \tag{3.4}$$

Furtherover for $s_c = 1$, *ie* for $p = 2^* - 1$ where $2^* = \frac{2N}{N-2}$ is the Sobolev exponent, the energy is left invariant by the scaling symmetry. Consequently the problem is energy critical.

The existence of type I blow up of ODE type is well known, but the existence of slow tyoep II blow up ie:

$$\underset{t\to T}{limitsup} \quad (T-t)^{\frac{1}{p-1}}\|u(t)\|_{L^\infty} = +\infty \tag{3.5}$$

has been a long standing open problem in the critical case. If p is subcritical in the Sobolev sense, it is well known by Giga and Kohn [8] that if u solution of (3.1) blows up at T, then

$$|u(t)|_{L^\infty} \leq C(T-t)^{\frac{-1}{p-1}} \quad \text{for} \quad t \in [0,T) \tag{3.6}$$

In [13] and [14], Matano and Merle prove the non existence of type II blow-up for $2^* - 1 < p < p_{JL}$, where p_{JL} is the coefficient of Joseph and Lundgren given by :

$$p_c = \begin{cases} +\infty & \text{for} \quad N \leq 10 \\ 1 + \frac{4}{N-4-2\sqrt{N-1}} & \text{for} \quad N \geq 11 \end{cases},$$

and for $p > p_{JL}$, it proves there exist type II blow up solutions [19], [20]. Even for the radial setting, the case $p = p_{JL}$ is still a challenging problem.

The case energy critical are formally study by Filippas, Herrero and Velàsquez in [7]. They give a detailed description of radially symmetric solutions, and predict a type II blow up in finite time for $N < 7$. An infinite set of rate of concentration is obtained for each dimension.

3.2. **Study the case four dimensional energy critical.** Let the problem

$$\partial_t u = \Delta u + u^3, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^4. \tag{3.7}$$

Let the Talenti-Aubin stationary solution

$$Q(r) = \frac{1}{1+\frac{r^2}{8}}. \tag{3.8}$$

which is the unique up to scaling radially symmetric solution to the stationary problem

$$\Delta Q + Q^3 = 0. \tag{3.9}$$

I prove in [25] the following theorem, which describe an type II blow up dynamics near the ground state.

Theorem 3.1 (Existence of type II blow up in dimension $N = 4$)**.** *Let Q be the Talenti Aubin soliton* (3.9)*. Then $\forall \alpha^* > 0$, there exists a radially symmetric initial data $u_0 \in H^1(\mathbb{R}^4)$ with*

$$E(Q) < E(u_0) < E(Q) + \alpha^* \tag{3.10}$$

such that the corresponding solution to the energy critical focusing parabolic equation (3.7) *blows up in finite time* $T = T(u_0) < \infty$ *in a type II regime according to the following dynamics: there exist* $u^* \in \dot{H}^1$ *such that:*

$$\nabla \left[u(t,x) - \frac{1}{\lambda(t)} Q\left(\frac{x}{\lambda(t)}\right)\right] \to \nabla u^* \quad in \quad L^2 \quad as \quad t \to T \tag{3.11}$$

at the speed

$$\lambda(t) = c(u_0)\,(1+o(1))\,\frac{T-t}{|\log(T-t)|^2} \quad as \quad t \to T \tag{3.12}$$

for some $c(u_0) > 0$. *Moreover, there hods the regularity of the asymptotic profile:*

$$\Delta u^* \in L^2 \tag{3.13}$$

To prove this result, I carry out the program which was implemented in [12] to adapt the study of the geometric wave equation in [21] to the semi linear cubic four dimensional wave equation. The main difficulty is the fact that the energy is non definite positive, and this induces a non positive eigenvalue in the spectrum of the linearized operator close to the ground state. This is the heart of the instability of the associated type II blow up scenario.

Natural directions of research are sill in dimension 4:

- classification of the flow near the ground state.
- Obtain, as in the article [24] for the harmonic heat flow, the set of speeds found by Filippas, Herrero and Velàsquez in [7].

3.3. **Other dimension.** The stationnary problem for $N \geq 3$

$$\Delta Q + Q^{\frac{N+2}{N-2}} = 0. \tag{3.14}$$

admits a unique radially symmetric solution, up to the scaling :

$$Q(r) = \left(\frac{1}{1+\frac{r^2}{N(N-2)}}\right)^{\frac{N-2}{2}}. \tag{3.15}$$

With the rescaled variables

$$\frac{ds}{dt} = \frac{1}{\lambda^2(t)}, \quad y = \frac{r}{\lambda(t)} \tag{3.16}$$

the problem (3.1) becomes

$$\partial_s u - \frac{\lambda_s}{\lambda}\Lambda u = \Delta u + u^{\frac{N+2}{N-2}} \tag{3.17}$$

in which the operator Λ is given by

$$\Lambda u = \frac{N-2}{2} u + y.\nabla u \tag{3.18}$$

To obtain our results, we begin with the construction of an approximate solution, given formally the law of $b(s)$, where we let

$$b(s) = -\frac{\lambda_s}{\lambda}\,(1+o(1))\,. \tag{3.19}$$

b is a positive function. This function must decrease towards 0 as s approaches infinity, if the solution blows up in finite time. The calculation for any dimension gives

$$b_s = \left(\frac{N}{2} - 3\right) b^2 + o(b^2) \tag{3.20}$$

The analysis of this ODE leads to the following:

Conjecture For $N > 6$, there is no type II blow up.

This problem is very interesting and would complete the Matano Merle analysis in super critical regimes. In a joint program with Frank Merle (Cergy, IHES), we hope to prove this conjecture. It also remains to extend the type II blow up for $N = 3, 5$, and here the case $N = 3$ displays some further technical difficulties due to the slow decay of the ground state.

4. **The Patlak-Keller-Segel model for chemotaxis**

4.1. **Introduction.** We study the two dimensional parabolic-elliptic Patlak-Keller-Segel model for chemotaxis:

$$(PKS) \quad \begin{cases} \partial_t u = \nabla \cdot (\nabla u + u \nabla \phi_u), \\ \phi_u = \frac{1}{2\pi} \log|x| \star u & (t,x) \in \mathbb{R} \times \mathbb{R}^2, \\ u_{|t=0} = u_0 > 0 \end{cases} \tag{4.1}$$

The corresponding non negative strong solution satisfies the conservation of mass

$$\int_{\mathbb{R}^2} u(t,x)dx = \int_{\mathbb{R}^2} u(0,x)dx \tag{4.2}$$

and the flow dissipates the logarithmically degenerate entropy:

$$\mathcal{F}(u) = \int u \log u + \frac{1}{2} \int u \phi_u \leq \mathcal{F}(u_0). \tag{4.3}$$

The scaling symmetry

$$u_\lambda(t,x) = \lambda^2 u(\lambda^2 t, \lambda x)$$

leaves the L^1 norm unchanged

$$\int_{\mathbb{R}^2} u_\lambda(t,x)dx = \int_{\mathbb{R}^2} u(\lambda^2 t, x)dx$$

and hence the problem is L^1 critical. Note from (4.3) that the problem is also *almost energy critical* and therefore particularly degenerate.

In [5], Carlen and Figalli prove that the ground state stationary solution

$$Q(r) = \frac{8}{(1+r^2)^2} \tag{4.4}$$

is up to symmetry the unique minimizer to the logarithmically degenerate entropy for u non negative such that $\int u = 8\pi = \int Q$. Remark that this entropy is non lower bounded for the other masses.

Blanchet, Dolbeault and Perthame describe in [4] the threshold effect for mass. For smooth well localized initial data, such as $\int u < \int Q$, the solution of (4.1) is global and zero is the universal local attractor. The case of critical mass with finite second moment is studied in [2], and the solution blows up in infinite time with formation of a Dirac mass. The case of critical mass with infinite second moment is studied in [3], and the solution converges to a soliton for a suitable norm.

Herrero and Velàsquez obtain in [11], [26] and [27] the first results on the formation of singularity for (4.1) in the radial case for super critical mass using the matched asymptotics techniques and an ODE approach.

4.2. **Results and perspectives.** Let the weighted L^2 space

$$\|\varepsilon\|_{L^2_Q} = \left(\int \frac{\varepsilon^2}{Q}\right)^{\frac{1}{2}} \tag{4.5}$$

and the weighted H^2 space:

$$\|\varepsilon\|_{H^2_Q} = \|\Delta\varepsilon\|_{L^2_Q} + \left\|\frac{\nabla\varepsilon}{1+|x|}\right\|_{L^2_Q} + \|\varepsilon\|_{L^2}. \tag{4.6}$$

We introduce the energy norm

$$\|\varepsilon\|_{\mathcal{E}} = \|\varepsilon\|_{H^2_Q} + \|\varepsilon\|_{L^1}. \tag{4.7}$$

We obtain in [23] the following:

Theorem 4.1 (Stable chemotactic blow up)**.** *There exists a set of initial data of the form*

$$u_0 = Q + \varepsilon_0 \in \mathcal{E}, \quad u_0 > 0, \quad \|\varepsilon_0\|_{\mathcal{E}} \ll 1$$

such that the corresponding solution $u \in \mathcal{C}([0,T), \mathcal{E})$ *to* (4.1) *satisfies the following:*
(i) Small super critical mass*:*

$$8\pi < \int u_0 < 8\pi + \alpha^*$$

for some $0 < \alpha^* \ll 1$ *which can be chosen arbitrarily small;*
(ii) Blow up *: the solution blows up in finite time* $0 < T < +\infty$*;*
(iii) Universality of the blow up bubble*: the solution admits for all times* $t \in [0,T)$ *a decomposition*

$$u(t,x) = \frac{1}{\lambda^2(t)}(Q+\varepsilon)\left(t, \frac{x}{\lambda(t)}\right)$$

with

$$\|\varepsilon(t)\|_{H^2_Q} \to 0 \quad as \quad t \to T \tag{4.8}$$

and the universal blow up speed:

$$\lambda(t) = \sqrt{T-t}e^{-\sqrt{\frac{|\log(T-t)|}{2}}+O(1)} \quad as \quad t \to T. \tag{4.9}$$

(iv) Stability*: the above blow up dynamics is stable by small perturbation of the data in* $\mathcal{E}$*:*

$$v_0 > 0, \quad \|v_0 - u_0\|_{\mathcal{E}} < \epsilon(u_0).$$

This result shows the robustness of our approach which enables us to attack non local problems. There are two main directions of research:

- Derivation of more non trivial dynamics near the ground state and possibly classification of the flow.
- Stability against non radial perturbations which is still confronted to some technical issues.

The parabolic-parabolic (KPS) system:

$$(PKS)\begin{cases} \partial_t u = \nabla.(\nabla u + u\nabla v), \\ \partial_t v = \Delta v - u, \\ u_{|t=0} = u_0 > 0, \\ v_{|t=0} = v_0 > 0 \end{cases} \quad (t,x) \in (\mathbb{R} \times \mathbb{R}^2) \tag{4.10}$$

displays a similar structure. The ground state for parabolic-elliptic model is still a stationary solution. In the super critical case $N = 3$, the existence of blow up solutions has been proved recently using a virial argument, but the existence of blow up solutions in the critical case is a long standing open problem. Using the dynamical approach to blow up and the functional setting developed in [23], I have obtained a result similar to the parabolic-elliptic case, which is in process of writing.

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