STABLE BLOW UP DYNAMICS FOR THE 1-COROTATIONAL ENERGY CRITICAL HARMONIC HEAT FLOW

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Abstract. We exhibit a stable finite time blow up regime for the 1-corotational energy critical harmonic heat flow from $\mathbb{R}^2$ into a smooth compact revolution surface of $\mathbb{R}^3$ which reduces to the semilinear parabolic problem

$$\partial_t u - \partial_r^2 u - \frac{\partial_r u}{r} + \frac{f(u)}{r^2} = 0$$

for a suitable class of functions f. The corresponding initial data can be chosen smooth, well localized and arbitrarily close to the ground state harmonic map in the energy critical topology. We give sharp asymptotics on the corresponding singularity formation which occurs through the concentration of a universal bubble of energy at the speed predicted in [2]. Our approach lies in the continuation of the study of the 1-equivariant energy critical wave map and Schrödinger map with $\mathbb{S}^2$ target in [22], [19].

1. Introduction

1.1. Setting of the problem. The harmonic heat flow between two embedded Riemanian manifolds $(N, g_N), (M, g_M)$ is the gradient flow associated to the Dirichlet energy of maps from $N \to M$:

$$\begin{cases} \partial_t v = \mathbb{P}_{T_v M}(\Delta_{g_N} v) \\ v_{|t=0} = v_0 \end{cases} \quad (t,x) \in \mathbb{R} \times N, \quad v(t,x) \in M \tag{1.1}$$

where $\mathbb{P}_{T_v M}$ is the projection onto the tangent space to M at v. The special case $N = \mathbb{R}^2$, $M = \mathbb{S}^2$ corresponds to the harmonic heat flow to the 2-sphere

$$\partial_t v = \Delta v + |\nabla v|^2 v, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2, \quad v(t,x) \in \mathbb{S}^2 \tag{1.2}$$

which appears in crystal physics and is related to the Landau Lifschitz equation of ferromagnetism, we refer to [2], [1], [8], [9] and references therein for a complete introduction to this class of problems. We shall from now on restrict our discussion to the case:

$$N = \mathbb{R}^2.$$

Local existence of solutions emanating from smooth data is well known. Note that the Dirichlet energy is dissipated by the flow

$$\frac{d}{dt}\left\{\int_{\mathbb{R}^2} |\nabla v|^2\right\} = -2\int_{\mathbb{R}^2} |\partial_t v|^2$$

and left invariant by the scaling symmetry

$$u_\lambda(t,x) = u(\lambda^2 t, \lambda x).$$

Hence the problem is *energy critical* and a singularity formation by energy concentration is possible. By the works of Struwe [25], Ding and Tian [6], Qing and Tian

[12] (see Topping [27] for a complete history of the problem), it is known that concentration implies the bubbling off of a non trivial harmonic map at a finite number of blow up points
$$v(t_i, a_i + \lambda(t_i)x) \to Q_i, \quad \lambda(t_i) \to 0 \tag{1.3}$$
locally in space. In particular, this shows the existence of a global in time flow on negatively curved targets where no nontrivial harmonic map exists.

1.2. **Corotational flows.** The existence of blow up solutions has been proved in various different geometrical settings, see in particular Chang, Ding, Ye [3], Coron and Ghidaglia [4], Qing and Tian [12], Topping [27]. We shall restrict in this paper onto flows with symmetries which are better understood.
Let a smooth closed curve in the plane parametrized by arclength
$$u \in [-\pi, \pi] \mapsto \left| \begin{array}{l} g(u) \\ z(u) \end{array} \right. , \quad (g')^2 + (z')^2 = 1,$$
where
$$(H) \quad \left\{ \begin{array}{l} g \in \mathcal{C}^\infty(\mathbb{R}) \ \text{ is odd and } 2\pi \text{ periodic,} \\ g(0) = g(\pi) = 0, \ \ g(u) > 0 \ \text{ for } \ 0 < u < \pi, \\ g'(0) = 1, \ \ g'(\pi) = -1, \end{array} \right. \tag{1.4}$$
then the revolution surface M with parametrization
$$(\theta, u) \in [0, 2\pi] \times [0, \pi] \mapsto \left| \begin{array}{l} g(u)\cos\theta \\ g(u)\sin\theta \\ z(u) \end{array} \right. ,$$
is a smooth[1] compact revolution surface of $\mathbb{R}^3$ with metric $(du)^2 + (g(u))^2(d\theta)^2$. Given a homotopy degree $k \in \mathbb{Z}^*$, the k-corotational reduction to (1.1) corresponds to solutions of the form
$$v(t,r) = \left| \begin{array}{l} g(u(t,r))\cos(k\theta) \\ g(u(t,r))\sin(k\theta) \\ z(u(t,r)) \end{array} \right. \tag{1.5}$$
which leads to the semilinear parabolic equation[2]:
$$\left\{ \begin{array}{l} \partial_t u - \partial_r^2 u - \frac{\partial_r u}{r} + k^2 \frac{f(u)}{r^2} = 0, \\ u_{t=0} = u_0 \end{array} \right. \quad f = gg'. \tag{1.6}$$
The k-corotational Dirichlet energy becomes
$$E(u) = \int_0^{+\infty} \left[|\partial_r u|^2 + k^2 \frac{(g(u))^2}{r^2} \right] rdr \tag{1.7}$$
and is minimized among maps with boundary conditions
$$u(0) = 0, \quad \lim_{r \to +\infty} u(r) = \pi \tag{1.8}$$
onto the least energy harmonic map Q_k which is the unique -up to scaling- solution to
$$r\partial_r Q_k = kg(Q_k) \tag{1.9}$$
satisfying (1.8), see for example [5].
In the case of $\mathbb{S}^2$ target $g(u) = \sin u$, the harmonic map is explicitly given by
$$Q_k(r) = 2\tan^{-1}(r^k). \tag{1.10}$$

[1] see eg [7]
[2] see (4.27)

In the series of works by Guan, Gustaffson, Tsai [8], Gustaffson, Nakanishi, Tsai [9], Q_k is proved to be *stable* by the flow (1.6) for $k \geq 3$, and in particular no blow up will occur near Q_k. Moreover, eternally oscillating solutions and infinite time blow up solutions are exhibited for $k = 2$.
We shall from now on and for the rest of the paper restrict attention to the degree $k = 1$ which generates the least energy harmonic map $Q \equiv Q_1$. For $\mathbb{D}^2$ initial manifold and $\mathbb{S}^2$ target, the formal analysis by Van den Berg, Hulshof and King [2] suggests through matching asymptotics the existence of a *stable generic* blow up regime with

$$u(t,r) \sim Q\left(\frac{r}{\lambda(t)}\right), \quad \lambda(t) \sim \frac{T-t}{|\log(T-t)|^2}.$$

In this direction, Angenent, Hulshof and Matano exhibit in [1] a class of corotational solutions which blow up in finite time with an estimate:

$$\lambda(t) = o(T-t) \quad \text{as} \quad t \to T.$$

The maximum principle plays an important role in this analysis. The sharp description of the singularity formation and in particular the understanding of the generic regime thus remain open.
More generally, let us recall that the derivation of the blow up speed for energy critical parabolic problems is poorly understood, and for example the derivation of sharp asymptotics of type II blow up for the energy critical semilinear problem

$$\partial_t u = \Delta u + u^{\frac{N+2}{N-2}}, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \quad N \geq 3$$

is open.

1.3. **Statement of the result.** Let $\mathcal{Q}$ the least energy harmonic map with degree 1 generated by the $Q \equiv Q_1$ solution to (1.9), explicitly:

$$\mathcal{Q}(x) = \left| \begin{array}{l} g(Q(t,r))\cos\theta \\ g(Q(t,r))\sin\theta \\ z(Q(t,r)) \end{array} \right. \tag{1.11}$$

For an integer $i \geq 1$, we let $\dot{H}^i$ be the completion of $\mathcal{C}^\infty_c(\mathbb{R}^2, \mathbb{R}^3)$ for the norm

$$\|v\|_{\dot{H}^i} = \|\Delta^{\frac{i}{2}} v\|_{L^2}.$$

We claim the existence and stability of a universal blow up regime emerging from 1-equivariant smooth data arbitrarily close to $\mathcal{Q}$ in the energy critical topology, together with sharp asymptotics on the singularity formation.

Theorem 1.1 (Stable blow up dynamics for the 1-corotational heat flow)**.** *Let $k = 1$ and g satisfy* (1.4)*. Let $\mathcal{Q}$ be the least energy harmonic map given by* (1.11)*. Then there exists an open set $\mathcal{O}$ of 1-corotational initial data of the form*

$$v_0 = \mathcal{Q} + \varepsilon_0, \quad \varepsilon_0 \in \mathcal{O} \subset \dot{H}^1 \cap \dot{H}^4$$

such that the corresponding solution $v \in \mathcal{C}([0,T), \dot{H}^1 \cap \dot{H}^4)$ to (1.1) *blows up in finite time $0 < T = T(v_0) < +\infty$ according to the following universal scenario:*
(i) Universality of the concentrating bubble*: there exists an asympotic profile $v^* \in \dot{H}^1$ and $\lambda \in \mathcal{C}^1([0,T), \mathbb{R}^*_+)$ such that*

$$\lim_{t \to T} \left\| v(t,x) - \mathcal{Q}\left(\frac{x}{\lambda(t)}\right) - v^* \right\|_{\dot{H}^1} = 0. \tag{1.12}$$

(ii) Sharp asymptotics*: the blow up speed is given by*

$$\lambda(t) = c(v_0)(1 + o(1))\frac{T-t}{|\log(T-t)|^2} \quad as \quad t \to T \tag{1.13}$$

for some $c(v_0) > 0$*.*
(iii) Regularity of the asymptotic profile*: there holds the additional regularity*

$$\Delta v^* \in L^2. \tag{1.14}$$

In other words, there exists a generic blow up regime with the law (1.13) as predicted in [2] for $g(u) = \sin u$, and blow up in this regime occurs by the concentration of a universal and quantized bubble of energy.

Comments on the result:

1. Energy method: Following the strategy developped in [15], [17], [21], [22], [19], our strategy of proof proceeds first through the construction of suitable approximate solutions, and then the control of the remaining radiation through a robust *energy method.* In particular, we make no use of the maximum principle, and hence we expect our strategy to be applicable to more complicated parabolic *systems* among which the full problem (1.1). Note also that parabolic problems cannot be solved backwards in time and involve smooth data. In this sense the construction of blow up solutions requires to follow the flow of smooth solutions forward in time and cannot be achieved by solving from blow up time for rough data as in [14], [11]. The set of initial data we construct in the proof of Theorem 1.1 contains compactly supported C^∞ 1-corotational functions.

2. Regularity of the asymptotic profile: The regularity of the asymptotic profile (1.14) is a completely new feature with respect to the regularity obtained in [18], [22] where the profile is just in the critical space. This would also allow one to quantify the convergence rate (1.12) and bound the error polynomially in time, which is a substantial improvement on the general convergence (1.3). This shows also the close relation between the blow up rate which is far above selfsimilarity[3] and the regularity of u^*, see [23] for related discussions, and explains formally why the problem under consideration should be thought of as "one derivative" above the wave map problem considered in [22].

3. Comparison with wave and Schrödinger maps: This result lies in the continuation of the works [22], [19] on the derivation of stable or codimension one blow up dynamics for the wave map:

$$\text{(WM)} \quad \left\{ \begin{array}{l} \partial_{tt}u - \Delta u = (|\partial_t u|^2 - |\nabla u|^2)u \\ u_{|t=0} = u_0, \quad \partial_t u_{|t=0} = u_1, \end{array} \right. \quad (t,x) \in \mathbb{R}\times\mathbb{R}^2, \quad u(t,x) \in \mathbb{S}^2, \tag{1.15}$$

and the Schrödinger map:

$$\text{(SM)} \quad \left\{ \begin{array}{l} u \wedge \partial_t u = \Delta u + |\nabla u|^2 u \\ u_{|t=0} = u_0, \end{array} \right. \quad (t,x) \in \mathbb{R}\times\mathbb{R}^2, \quad u(t,x) \in \mathbb{S}^2, \tag{1.16}$$

in both cases from $\mathbb{R}\times\mathbb{R}^2 \to \mathbb{S}^2$. For the wave map, a stable blow up dynamics within the k-corotational symmetry class (1.5) is exhibited in [22] for all homotopy number $k \geq 1$ with an almost self similar blow up speed, see also [24]. In [19],

[3]corresponding to the law $\lambda(t) \sim \sqrt{T-t}$.

the Schrödinger map problem is considered within the class of functions with 1-equivariant symmetry ie of the form

$$v(t,x)=e^{\theta R}w(t,r),\quad w(t,r)=\left|\begin{array}{l} w_1(t,r)\\ w_2(t,r)\\ w_3(t,r)\end{array}\right. \tag{1.17}$$

with

$$R=\left(\begin{array}{lll} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{array}\right),\quad e^{\theta R}=\left(\begin{array}{lll} \cos(\theta) & -\sin(\theta) & 0\\ \sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 1\end{array}\right). \tag{1.18}$$

A codimension one set of smooth initial data is exhibited for which concentration occurs

$$\lim_{t\to T}\left\|v(t,x)-e^{\Theta^*R}\mathcal{Q}\left(\frac{x}{\lambda(t)}\right)-v^*\right\|_{\dot{H}^1}=0 \tag{1.19}$$

for some $\Theta^*\in\mathbb{R}$, $v^*\in\dot{H}^1$ at the speed given by (1.13):

$$\lambda(t)=c(v_0)(1+o(1))\frac{T-t}{|\log(T-t)|^2}.$$

Note that the 1-equivariant symmetry is also preserved by the wave map and the harmonic heat flow (1.2), and the 1-corotational symmetry[4] (1.5) corresponds to the 1-equivariant symmetry (1.17) with $w_2\equiv 0$, and hence such maps are *not allowed to rotate around the* e_z *axis*. This extra degree of freedom in 1-equivariant symmetry is shown in [19] to stabilize the system and leads to a codimension one blow up phenomenon for the Schrödinger map. We expect the same phenomenon to occur here, and we conjecture that the blow up solutions constructed in Theorem 1.1 correspond to a codimension one phenomenon for the full problem (1.2).

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Notations: We introduce the differential operator

$$\Lambda f=y\cdot\nabla f\quad\text{(energy critical scaling)}.$$

Given a positive number $b>0$, we let

$$B_0=\frac{1}{\sqrt{b}},\quad B_1=\frac{|\log b|}{\sqrt{b}}. \tag{1.20}$$

Given a parameter $\lambda>0$, we let

$$u_\lambda(r)=u(y)\quad\text{with}\quad y=\frac{r}{\lambda}.$$

We let χ be a positive nonincreasing smooth cut off function with

$$\chi(y)=\left\{\begin{array}{ll} 1 & \text{for}\ \ y\le 1,\\ 0 & \text{for}\ \ y\ge 2.\end{array}\right.$$

Given a parameter $B>0$, we will denote:

$$\chi_B(y)=\chi\left(\frac{y}{B}\right). \tag{1.21}$$

[4]which is not preserved by the Schrödinger map.

We shall systematically omit the measure in all radial two dimensional integrals and note:

$$\int f = \int_0^{+\infty} f(r)rdr.$$

1.4. **Strategy of the proof.** Let us give a brief insight into the strategy of the proof of Theorem 1.1 which adapts the strategy developed in [22], [19].

(i). Formal derivation of the law
Let us look for a modulated solution $u(t,r)$ of (1.6) in renormalized form

$$u(t,r) = z(s,y), \quad y = \frac{r}{\lambda(t)}, \quad \frac{ds}{dt} = \frac{1}{\lambda^2(t)}$$

which leads to the self similar equation:

$$\partial_s z - \Delta z + b\Lambda z + \frac{f(z)}{y^2} = 0, \quad b = -\frac{\lambda_s}{\lambda}. \tag{1.22}$$

The speed of concentration $\lambda(t)$ is an unknown of the problem, but we know a priori from energy constraint that the renormalized flow $z(s,y)$ must evolve close to the harmonic map $Q(y)$. This forces in particular the drift $b = -\frac{\lambda_s}{\lambda}$ to be *uniformly small in time*. We therefore proceed to a slow variable expansion[5] and look for a solution to (1.22) of the form:

$$z(s,y) = Q(y) + b(s)T_1(y) + b^2(s)T_2(y) + \dots.. \tag{1.23}$$

and where the time dependance appears only in the modulation function b through the ODE

$$b_s(s) = -c_2b^2(s) + c_3b^3(s) + \dots \tag{1.24}$$

Here the profiles $T_1(y), T_2(y)\dots$ and constants $c_2, c_3, \dots$ are unkwnowns to be determined. Plugging in the expansion (1.23), (1.24) into (1.22) and matching the powers of b yields at order b:

$$HT_1 + \Lambda Q = 0 \tag{1.25}$$

and at order b^2:

$$HT_2 + \Lambda T_1 - c_2T_1 + \frac{f''(Q)}{y^2}T_1^2 = 0. \tag{1.26}$$

Here H is the linearized Hamiltonian close to Q ie the Schrödinger operator

$$H = -\Delta + \frac{f'(Q)}{y^2}.$$

It displays a remarkable algebraic structure[6], and is in particular nonnegative with a resonance[7] generated by the energy critical scaling symmetry:

$$H(\Lambda Q) = 0$$

where from explicit computation[8]

$$\Lambda Q \sim \frac{c}{y} \text{ as } y \to \infty \text{ and thus } \Lambda Q \notin L^2. \tag{1.27}$$

[5]as standard in various settings, for example geometric optics, see for example [26].

[6]see section 2.2.

[7]ie a non L^2 eigenstate for the zero eigenvalue.

[8]Note that this slow decay is intrinsic to the $k = 1$ degree case, as higher degree yield faster decay, see (1.10).

The Green's functions of H are explicit, and a smooth solution to (1.25) can be computed which displays the *growth* at infinity:

$$T_1(y) \sim y\log y + cy + O\left(\frac{|\log y|^2}{y}\right) \quad \text{as} \quad y \to +\infty.$$

We now aim at solving (1.26) with the *least possible growth* for T_2. The choice $c_2 = 1$ cancels the worst $y\log y$ growth:

$$\Lambda T_1 - T_1 = y + O\left(\frac{|\log y|^2}{y}\right) \tag{1.28}$$

which after inversion of (1.26) leads to a growth:

$$T_2(y) \sim y^3 \quad \text{as} \quad y \to +\infty. \tag{1.29}$$

An essential feature here is that an additional *logarithmic gain* can be obtained by manufacturing an expansion

$$b_s = -b^2\left(1 + \frac{2}{|\log b|}\right) \tag{1.30}$$

which leads to an improved control after the parabolic zone:

$$T_2(y) = O\left(\frac{y}{b|\log b|}\right) \ll y^3 \quad \text{for} \quad y \geq \frac{1}{\sqrt{b}}. \tag{1.31}$$

Similar lower order corrections in the law for b like (1.30) occured in [20], [17], [22], [19] and can be derived from a *flux type computation*, or equivalently a boundary term in energy critical Pohozaev type integration by parts generated by the explicit growth (1.28). Here the gain is very explicit and relies on the direct introduction of the *radiative correction*, see step 3 of the proof of Proposition 2.4 and Remark 2.7. The above heuristical argument thus leads to the system of ODE's:

$$-\frac{\lambda_s}{\lambda} = b, \quad b_s = -b^2\left(1 + \frac{2}{|\log b|}\right), \quad \frac{ds}{dt} = \frac{1}{\lambda^2}. \tag{1.32}$$

We integrate this with initial data

$$0 < b(0) \ll 1, \quad \lambda(0) > 0$$

and easily show that the corresponding scaling parameter $\lambda(t)$ touches zero in finite time $0 < T = T(b(0), \lambda(0)) < +\infty$, this is blow up, at a rate:

$$\lambda(t) = c_{b_0,\lambda_0}(1 + o(1))\frac{T - t}{|\log(T - t)|^2} \quad \text{as} \quad t \to T,$$

this is (1.13).

(ii). Decomposition of the flow and modulation equations
Let then

$$Q_{b(s)}(y) = Q(y) + b(s)T_2(y) + b^2(s)T_2(y)\dots$$

a sufficiently high order correction to (1.22) in the regime (1.24) measured by the error:

$$\Psi_b = -b^2(T_1 + 2bT_2) - \Delta Q_b + b\Lambda Q_b + \frac{f(Q_b)}{y^2}$$

which is formally cubic in b, and consider an initial data of the form

$$u_0(y) = Q_{b_0}(y) + \varepsilon_0(y), \quad |\varepsilon_0(y)| \ll b_0 \ll 1$$

in some suitable sense. We introduce a modulated decomposition of the flow

$$u(t,r) = (Q_{b(t)} + \varepsilon)\left(t, \frac{r}{\lambda(t)}\right) \tag{1.33}$$

where the modulation parameters $(b(t), \lambda(t))$ are chosen in order to manufacture the orthogonality conditions:

$$(\varepsilon(t), \Phi_M) = (\varepsilon(t), H\Phi_M) = 0. \tag{1.34}$$

Here $\Phi_M(y)$ is some fixed direction depending on some large constant M which generates an approximation of *the kernel*[9] *of* H^2, see (3.5). This orthogonal decomposition, which for each fixed time t directly follows from the implicit function theorem, now allows us to compute the modulation equations governing the parameters $(b(t), \lambda(t))$. The Q_b construction is precisely manufactured to produce the expected ODE's[10]:

$$\left|\frac{\lambda_s}{\lambda} + b\right| + \left|b_s + b^2\left(1 + \frac{2}{|\log b|}\right)\right| \ll \|\varepsilon\|_{loc} + b^2\left(1 + \frac{2}{|\log b|}\right) \tag{1.35}$$

where $\|\varepsilon\|_{loc}$ measures a *local in space* interaction with the harmonic map:

$$\|\varepsilon\|^2_{loc} \sim \int \frac{|\varepsilon|^2}{1+y^8}.$$

(iii). Control of radiation and monotonicity formula.

We now need to show that *in the regime we are focusing on*, the radiation ε does not perturb the expected dynamical system (1.32), and from (1.35) this will follow from a bound:

$$\int \frac{|\varepsilon|^2}{1+y^8} \lesssim \frac{b^4}{|\log b|^2}. \tag{1.36}$$

In other words, the infinite dissipative dynamics of the radiation ε is fully controlled by the one dimensional projection b, and does not perturb the leading order dynamics. Controlling radiation is the heart of the analysis. Related bounds were derived in [9] for $k \geq 2$ using parabolic propagator estimates which become more delicate in the presence of resonances and with slow decay of the harmonic map, and we use here the approach developped in [22], [23]. The idea is to rewrite the decomposition (1.33) in original non rescaled variables as

$$u(t,r) = Q_{b(t)}\left(t, \frac{r}{\lambda(t)}\right) + w(t,r), \quad w(t,r) = \varepsilon\left(t, \frac{r}{\lambda(t)}\right). \tag{1.37}$$

The equation for $w(t,r)$ is roughly of the form[11]

$$\partial_t w + H_{\lambda(t)} w = \frac{1}{\lambda^2}\Psi_{b(t)}\left(\frac{r}{\lambda}\right) + \text{(lower order terms)} \tag{1.38}$$

where H_λ denotes the linearized operator close to the focusing harmonic map:

$$H_\lambda = -\Delta + \frac{f'(Q_\lambda)}{r^2}, \quad Q_\lambda(r) = Q\left(\frac{r}{\lambda}\right).$$

We now perform an energy type identity on (1.38) and aim at controlling Sobolev type norms on w build on iterates of the Hamiltonian H_λ. The difficulty is twofold: first we need to treat the terms induced by the time dependance of the Hamiltonian H_λ, and here we use mostly the algebraic structure of H and*dissipation*; second we

[9]Smooth solutions to $H^2 u = 0$ lie in Span$(\Lambda Q, T_1)$ and therefore decay badly near $+\infty$, hence the need for a localized approximation provided by Φ_M.

[10]see Lemma 3.2.

[11]see (3.27).

need to estimate suitable weighted Sobolev norms of Ψ_b. Here let us insist onto the fact that Ψ_b contains a priori high order terms in b against functions which grow in y, and hence it will be small in weighted norms only as expressed by Proposition 2.4. The energy bound we derive relies on a monotonicity formula which takes the form of the Sobolev $\dot{H}^4$ bound -see Proposition 3.4 for a precise statement-:

$$\frac{d}{dt}\|H^2_\lambda w\|^2_{L^2} \lesssim \frac{b}{\lambda^8}\left[\frac{b^4}{|\log b|^2} + (\text{lower order terms})\right]. \tag{1.39}$$

(iv). Closing the bootstrap.

We now integrate in time the monotonicity bound (1.39) assuming that the parameters follow to leading order the regime (1.32). This first yields using the rescaling (1.37):

$$\|H^2\varepsilon(t)\|^2_{L^2} \lesssim \left(\frac{\lambda(t)}{\lambda(0)}\right)^6 \|H^2\varepsilon(0)\|^2_{L^2} + \lambda^6(t)\int_0^t \frac{b^5(\tau)}{\lambda^8(\tau)|\log b(\tau)|^2}d\tau.$$

We next integrate the ODE (1.32) to derive the bounds:

$$\lambda^6(t) \lesssim b^5(t), \quad \lambda^6(t)\int_0^t \frac{b^5(\tau)}{\lambda^8(\tau)|\log b(\tau)|^2}d\tau \lesssim \frac{b^4(t)}{|\log b(t)|^2}$$

and finally obtain the pointwise bound:

$$\|H^2\varepsilon(t)\|^2_{L^2} \lesssim \frac{b^4(t)}{|\log b(t)|^2}.$$

We now observe that *provided we removed a suitable localization of the kernel of* H^2, this is the choice of orthogonality conditions (1.34), H^2 is a coercive operator which satisfies weighted Hardy type bounds, see Lemma B.2:

$$\int \frac{|\varepsilon|^2}{1+y^8} \lesssim \|H^2\varepsilon\|^2_{L^2} \lesssim \frac{b^4}{|\log b|^2} \tag{1.40}$$

and (1.36) is proved.

The above *linear* argument paves the way to close bootstrap bounds for $(b, \lambda, \varepsilon)$ in the decomposition (1.33). The remaining standard difficulty is to control the *nonlinear terms* in particular in the monotonicity estimate (1.39). This involves some technicalities including interpolation bounds build on the Hardy bounds (1.40), see Lemma B.3, and a careful treatment of both the singularity of the nonlinearity at the origin induced by the corotational change of variables, and the slow decay of functions at infinity. The analysis requires a careful track of polynomial weights both at the origin and infinity in the Hardy type estimates we use.

Finally, let us observe that the key in the above proof is to provide a sufficiently small error $\frac{b^4}{|\log b|^2}$ in the right hand side of (1.39), and this error is from (1.38) directly related to the size of Ψ_b in suitable norms, and hence to the order of expansion in terms of power of b in the construction of Q_b. An explicit computation reveals that we need to construct at least the first three terms in the Q_b expansion according to (2.18), see Remark 2.6. Moreover, from direct computations again, the dominant terms in the size of Ψ_b leave in the parabolic zone $y \sim \frac{1}{\sqrt{b}}$ where improved decay bounds from (1.29) to (1.31) induced by the introduction of suitable radiative corrections will turn out to be essential[12].

This paper is organized as follows. In section 2, we construct the aprroximate self similar solutions Q_b and obtain sharp estimates on the error term Ψ_b. In section

[12]see in particular the proof of (2.23).

3, we set up the bootstrap argument, Proposition 3.1, and derive the fundamental Sobolev $\dot{H}^4$ control, Proposition 3.4, which is the heart of the analysis. In section 4, we close the bootstrap bounds which easily imply the blow up statement of Theorem 1.1. Some further slight improvement of the obtained bounds yields the regularity (1.14) of the asymptotic profile.

2. Construction of the approximate profile

We follow the scheme of proof in [22], [19] and proceed in this section with the construction of suitable approximate solutions to the renormalized flow (1.22).

2.1. Asymptotics of the 1-corotational harmonic map.

Let us start with recalling the structure of the harmonic map Q in the context of (1.6) which is the unique -up to scaling- solution to

$$\Lambda Q = g(Q), \quad Q(0)=0, \quad \lim_{r\to+\infty} Q(r) = \pi. \tag{2.1}$$

This equation can be integrated explicitly and leads to the following asymptotics:

Lemma 2.1 (Asymptotics of the harmonic map)**.** *There holds $Q\in\mathcal{C}^\infty([0,+\infty),[0,\pi))$ with the Taylor expansions*[13]*:*

$$Q(y) = \Sigma_{i=0}^{p} c_i y^{2i+1} + O(y^{2p+3}) \quad as \quad y\to 0, \quad c_0\neq 0, \tag{2.2}$$

$$Q(y) = \pi - \frac{2}{y} - \Sigma_{i=1}^{p}\frac{d_i}{y^{2i+1}} + O\left(\frac{1}{y^{2p+3}}\right) \quad as \quad y\to+\infty. \tag{2.3}$$

Remark 2.2. Recall that (2.1) is subject to the scaling symmetry $Q(y)\to Q(\lambda y)$, $\lambda>0$. The scaling constant $\lambda>0$ affects the decay of the map near $+\infty$, and we define Q to be the unique solution with from (2.3):

$$\Lambda Q(y) \sim \frac{2}{y} \quad \text{for} \quad y\to+\infty.$$

This arbitrary choice has been made to match the more familiar case of the round sphere:

$$Q(y) = 2\tan^{-1}(y), \quad \Lambda Q(y) = \frac{2y}{1+y^2} \sim \frac{2}{y} \quad \text{as} \quad y\to+\infty.$$

Proof of Lemma 2.1 From the Taylor expansion of g at 0 and π given by (1.4), we have:

$$\int_a^\pi \frac{\pi-\tau-g(\tau)}{(\pi-\tau)g(\tau)}d\tau + \log(\pi-a) \to \left\{\begin{array}{ll} +\infty & \text{as} \quad a\to 0 \\ -\infty & \text{as} \quad a\to\pi \end{array}\right.$$

and we may thus find $a\in(0,\pi)$ such that

$$\int_a^\pi \frac{\pi-\tau-g(\tau)}{(\pi-\tau)g(\tau)}d\tau + \log(\pi-a) = \log 2. \tag{2.4}$$

We then let

$$G(u) = \int_a^u \frac{d\tau}{g(\tau)} \tag{2.5}$$

which from (1.4) is a diffeomorphism from $(0,\pi)$ onto $(-\infty,+\infty)$. Let now Q be the normalized solution to (2.1) given by

$$Q(y) = G^{-1}(\log y), \quad y\in[0,+\infty). \tag{2.6}$$

[13] up to scaling

We compute near π from the normalization (2.4):

$$\begin{aligned} G(u) &= \int_a^u \frac{d\tau}{g(\tau)} = \int_a^u \frac{\pi-\tau-g(\tau)}{(\pi-\tau)g(\tau)}d\tau + \int_a^u \frac{d\tau}{\pi-\tau} \\ &= \int_a^\pi \frac{\pi-\tau-g(\tau)}{(\pi-\tau)g(\tau)}d\tau + \log(\pi-a) - \log(\pi-u) - \int_u^\pi \frac{\pi-\tau-g(\tau)}{(\pi-\tau)g(\tau)}d\tau \\ &= -\log\left(\frac{\pi-u}{2}\right) + \Sigma_{i=1}^p \tilde{d}_i(\pi-u)^{2i} + O\left((\pi-u)^{2p+2}\right) \quad \text{as} \quad u\to\pi \end{aligned} \tag{2.7}$$

and near the origin:

$$\begin{aligned} G(u) &= \log u - \log a + \int_0^u \frac{\tau-g(\tau)}{\tau g(\tau)}d\tau - \int_0^a \frac{\tau-g(\tau)}{\tau g(\tau)}d\tau \\ &= \log u + \tilde{c}_0 + \Sigma_{i=1}^p \tilde{c}_i u^{2i} + O\left(u^{2p+2}\right) \quad \text{as} \quad u\to 0, \end{aligned}$$

and these developments together with (2.6) now yield (2.2), (2.3). This concludes the proof of Lemma 2.1.

2.2. **The linearized Hamiltonian.** We recall in this section the structure of the linearized operator close to Q. Let the potentials

$$Z = g'(Q), \quad V = Z^2 + \Lambda Z = f'(Q), \quad \widetilde{V} = (1+Z)^2 - \Lambda Z, \tag{2.8}$$

then the linearized operator close to Q is the Schrödinger operator:

$$H = -\Delta + \frac{V}{y^2}. \tag{2.9}$$

As observed in [24], an important consequence of the Bogomolny'i's factorization of the Dirichlet energy (1.7) is the decomposition

$$H = A^*A$$

where A^* is the adjoint of A with respect to the $L^2(\mathbb{R}^2)$ scalar product, explicitly:

$$A = -\partial_y + \frac{Z}{y}, \quad A^* = \partial_y + \frac{1+Z}{y}.$$

The kernels of A and A^* on $\mathbb{R}_+^*$ are explicit:

$$Au = 0 \ \text{ iff } \ u \in \text{Span}(\Lambda Q), \quad A^*u = 0 \ \text{ iff } \ u \in \text{Span}\left(\frac{1}{y\Lambda Q}\right), \tag{2.10}$$

and thus the kernel of H on $\mathbb{R}_+^*$ is:

$$Hu = 0 \ \text{ iff } \ u \in \text{Span}(\Lambda Q, \Gamma) \tag{2.11}$$

with

$$\Gamma(y) = \Lambda Q \int_1^y \frac{dx}{x(\Lambda Q(x))^2} = \begin{cases} O(\frac{1}{y}) & \text{as } \ y \to 0, \\ \frac{y}{4} + O\left(\frac{\log y}{y}\right) & \text{as } \ y \to +\infty. \end{cases} \tag{2.12}$$

In particular, H is a positive operator on $\dot{H}^1_{rad}$ with a *resonance* ΛQ at the origin induced by the energy critical scaling invariance. We also introduce the conjugate Hamiltonian

$$\tilde{H} = AA^* = -\Delta + \frac{\widetilde{V}}{y^2} \tag{2.13}$$

which is definite positive by construction and (2.10), see Lemma B.1. Finally, let us compute using (1.4), (2.2), (2.3) the behavior of $Z, V, \widetilde{V}$ at 0 and $+\infty$ which will be fundamental in our analysis:

$$Z(y) = \left\{ \begin{array}{l} 1 + \Sigma_{i=1}^p c_i y^{2i} + O(y^{2p+2}) \quad \text{as} \quad y \to 0, \\ -1 + \Sigma_{i=1}^p \frac{c_i}{y^{2i}} + O\left(\frac{1}{y^{2p+2}}\right) \quad \text{as} \quad y \to +\infty, \end{array} \right. \tag{2.14}$$

$$V(y) = \left\{ \begin{array}{l} 1 + \Sigma_{i=1}^p c_i y^{2i} + O(y^{2p+2}) \quad \text{as} \quad y \to 0, \\ 1 + \Sigma_{i=1}^p \frac{c_i}{y^{2i}} + O\left(\frac{1}{y^{2p+2}}\right) \quad \text{as} \quad y \to +\infty, \end{array} \right. \tag{2.15}$$

$$\widetilde{V}(y) = \left\{ \begin{array}{l} 4 + \Sigma_{i=1}^p c_i y^{2i} + O(y^{2p+2}) \quad \text{as} \quad y \to 0, \\ \Sigma_{i=1}^p \frac{c_i}{y^{2i}} + O\left(\frac{1}{y^{2p+2}}\right) \quad \text{as} \quad y \to +\infty, \end{array} \right. \tag{2.16}$$

where $(c_i)_{i\geq 1}$ stands for some generic sequence of constants which depend on the Taylor expansion of g at 0 and π.

Remark 2.3. The exact values for the $\mathbb{S}^2$ target $g(u) = \sin u$ are given by:

$$Z(y) = \frac{1-y^2}{1+y^2}, \quad V(y) = \frac{y^4 - 6y^2 + 1}{(1+y^2)^2}, \quad \widetilde{V}(y) = 2(1+Z) = \frac{4}{1+y^2}.$$

2.3. **Slowly modulated approximate profiles.** We now aim at constructing a suitable approximate solution to (1.22) near the harmonic map Q with *moderate growth* as $y \to +\infty$ by adapting the slowly modulated ansatz approach developed in [16], [10], [22], [19]. We will see that this naturally leads to the leading order modulation equation[14]:

$$b_s = -b^2(1 + o(1)). \tag{2.17}$$

Proposition 2.4 (Construction of the approximate profile)**.** *Let $M > 0$ be a large enough universal constant. Then there exists $0 < b^*(M) \ll 1$ such for all $0 < b < b^*(M)$, there exist smooth profiles $(T_i(y,b))_{1\leq i\leq 3}$ such that the following holds true. Let the approximate profile*

$$Q_b(y) = Q(y) + bT_1(b,y) + b^2T_2(b,y) + b^3T_3(b,y) = Q(y) + \alpha(b,y) \tag{2.18}$$

and define the error:

$$\Psi_b = -b^2(T_1 + 2bT_2) - \Delta Q_b + b\Lambda Q_b + \frac{f(Q_b)}{y^2}, \tag{2.19}$$

then there holds the bounds:
(i) Weighted bounds:

$$\int_{y\leq 2B_1} |H\Psi_b|^2 \lesssim b^4 |\log b|^2, \tag{2.20}$$

$$\int_{y\leq 2B_1} \frac{1+|\log y|^2}{1+y^4} |H\Psi_b|^2 \lesssim \frac{b^4}{|\log b|^2}, \tag{2.21}$$

$$\int_{y\leq 2B_1} \frac{1+|\log y|^2}{1+y^2} |AH\Psi_b|^2 \lesssim \frac{b^5}{|\log b|^2}, \tag{2.22}$$

$$\int_{y\leq 2B_1} |H^2\Psi_b|^2 \lesssim \frac{b^6}{|\log b|^2} \tag{2.23}$$

(ii) Flux computation: Let Φ_M be given by (3.5), then:

$$\frac{(H\Psi_b, \Phi_M)}{(\Lambda Q, \Phi_M)} = -\frac{2b^2}{|\log b|} + O\left(\frac{b^2}{|\log b|^2}\right). \tag{2.24}$$

[14] see Remark 2.7

Remark 2.5. From the proof, the profiles T_i are smooth and display some explicit algebraic growth near $y \to +\infty$, see the precise bounds (2.30), (2.31), (2.45), (2.50). Also the dependence on b of the profiles T_i is lower order, and we shall in the sequel omit this dependence and note

$$T_i(y), \alpha(y)$$

to ease notations. Observe that the decomposition (2.18) in the regime (2.17) leads to leading order to:

$$\partial_s Q_b - \Delta Q_b + b\Lambda Q_b + \frac{f(Q_b)}{y^2} = \Psi_b + (\text{lower order terms})$$

and hence Q_b is a high order approximate solution to the renormalized flow (1.22) in the regime (2.17). The flux computation (2.24) is the heart of the logarithmic correction to the ODE $b_s = -b^2$ and will lead to the correct modulation equation (1.30), see (3.32).

Remark 2.6. The key here is the estimate (2.23) which is the leading order term driving the energy bound (3.4). This high order error requires an order three expansion of the profile in (2.18).

Proof of Proposition 2.4

We recall that given a small parameter $0 < b \ll 1$, we define the large scales B_0, B_1 given by (1.20). B_0 is the boundary of the parabolic zone $y \lesssim \frac{1}{\sqrt{b}}$ where most estimates will take place, and B_1 is a suitable size after which the profiles T_i cease to make sense in the dynamical description of the solution, and hence we will proceed to localization of the profiles for $y \geq 2B_1$ in Proposition 2.8. We therefore need a control of T_i, Ψ_b in the zone $y \leq 2B_1$ only.

step 1 Expansion in powers of b.

Let us compute the error (2.19) for a general decomposition (2.18) using a Taylor expansion:

$$\begin{aligned} f(Q_b) &= f(Q) + b\left[T_1 f'(Q)\right] + b^2\left[T_2 f'(Q) + \frac{1}{2}T_1^2 f''(Q)\right] \\ &+ b^3\left[T_3 f'(Q) + T_1T_2 f''(Q) + \frac{1}{6}f^{(3)}(Q)T_1^3\right] + R_1 + R_2 \end{aligned}$$

with

$$R_1 = \frac{1}{2}f''(Q)\left[\alpha^2 - b^2T_1^2 - 2b^3T_1T_2\right] + \frac{1}{6}f^{(3)}(Q)\left[\alpha^3 - b^3T_1^3\right], \tag{2.25}$$

$$R_2 = \frac{\alpha^4}{6}\int_0^1 (1-\tau)^3 f^{(4)}(Q+\tau\alpha)d\tau. \tag{2.26}$$

Hence from (2.19):

$$\begin{aligned} \Psi_b &= b\left(HT_1 + \Lambda Q\right) \\ &+ b^2\left(HT_2 - T_1 + \Lambda T_1 + \frac{f''(Q)}{2y^2}T_1^2\right) \\ &+ b^3\left(HT_3 - 2T_2 + \Lambda T_2 + \frac{f''(Q)}{y^2}T_1T_2 + \frac{1}{6}\frac{f^{(3)}(Q)}{y^2}T_1^3\right) \\ &+ b^4\Lambda T_3 + \frac{1}{y^2}\left[R_1 + R_2\right]. \end{aligned} \tag{2.27}$$

Step 2 Construction of T_1.

We may invert H explicitly from (2.11) and a smooth solution at the origin to $Hu=-f$ is given by:

$$u=\Gamma(y)\int_0^y f\Lambda Q x dx-\Lambda Q(y)\int_0^y f\Gamma x dx. \tag{2.28}$$

Observe that if f admits the Taylor expansion at the origin

$$f(y)=c_1y+c_3y^3+O(y^5),$$

then

$$u(y)=d_3y^3+O(y^5)\quad\text{as}\quad y\to 0. \tag{2.29}$$

Indeed, the Wronskian relation $\Gamma'(\Lambda Q)-(\Lambda Q)'\Gamma=\frac{1}{y}$ implies

$$A\Gamma=-\Gamma'+\frac{Z}{y}\Gamma=-\Gamma'+\frac{(\Lambda Q)'}{\Lambda Q}\Gamma=-\frac{1}{y\Lambda Q}$$

from which using $A\Lambda Q=0$:

$$Au=A\Gamma\int_0^y f\Lambda Q x dx=-\frac{1}{y\Lambda Q}\int_0^y f\Lambda Q x dx=c_2y^2+c_4y^4+O(y^6)$$

near the origin for some constants (c_2,c_4). We now integrate using $u=O(y^3)$ at the origin from (2.28) and thus:

$$u=-\Lambda Q\int_0^y\frac{Au}{\Lambda Q}dx=d_3y^3+O(y^5).$$

We now let T_1 be the solution to $HT_1+\Lambda Q=0$ given by

$$T_1(y)=\Gamma(y)\int_0^y(\Lambda Q)^2xdx-\Lambda Q(y)\int_0^y\Lambda Q\Gamma x dx.$$

We compute from Lemma 2.1 and (2.28), (2.29) the behavior:

$$T_1(y)=\left\{\begin{array}{l} y\text{log}y+e_0y+O\left(\frac{(\text{log}y)^2}{y}\right)\quad\text{as}\quad y\to+\infty\\ d_3y^3+O(y^5)\quad\text{as}\quad y\to 0\end{array}\right. \tag{2.30}$$

for some universal constant e_0, and similarly:

$$\Lambda^iT_1(y)=\left\{\begin{array}{l} y\text{log}y+(e_0+i)y+O\left(\frac{(\text{log}y)^2}{y}\right)\quad\text{as}\quad y\to+\infty\\ d_{3,i}y^3+O(y^5)\quad\text{as}\quad y\to 0\end{array}\right.\quad\text{for}\quad 1\le i\le 3. \tag{2.31}$$

Note that T_1 does not depend on b:

$$\frac{\partial T_1}{\partial b}=0.$$

Step 3 Construction of the radiation Σ_b.

We now construct the radiation term which will allow us to capture further cancellation near the parabolic zone $y\sim\frac{1}{\sqrt{b}}$, see Remark 2.7. Recall the definition (1.20) and let:

$$c_b=\frac{4}{\int\chi_{\frac{B_0}{4}}(\Lambda Q)^2}=\frac{2}{|\text{log}b|}\left(1+O(\frac{1}{|\text{log}b|})\right) \tag{2.32}$$

and

$$d_b=c_b\int_0^{B_0}\chi_{\frac{B_0}{4}}\Lambda Q\Gamma y dy=\frac{C}{b|\text{log}b|}\left(1+O\left(\frac{1}{|\text{log}b|}\right)\right) \tag{2.33}$$

Let Σ_b be the solution to

$$H\Sigma_b = -c_b\chi_{\frac{B_0}{4}}\Lambda Q + d_b H[(1-\chi_{3B_0})\Lambda Q] \tag{2.34}$$

given by

$$\Sigma_b(y) = \Gamma(y)\int_0^y c_b\chi_{\frac{B_0}{4}}(\Lambda Q)^2 xdx - \Lambda Q(y)\int_0^y c_b\chi_{\frac{B_0}{4}}\Gamma\Lambda Q xdx + d_b(1-\chi_{3B_0})\Lambda Q(y) \tag{2.35}$$

Observe that by definition :

$$\Sigma_b = \begin{cases} c_bT_1 \quad \text{for} \quad y \leq \frac{B_0}{4} \\ \\ 4\Gamma \quad \text{for} \quad y \geq 6B_0. \end{cases} \tag{2.36}$$

We now estimate for $6B_0 \leq y \leq 2B_1$:

$$\Sigma_b(y) = y + O(\frac{\log y}{y}) \quad \Lambda\Sigma_b(y) = y + O(\frac{\log y}{y}) \tag{2.37}$$

and for $y \leq 6B_0$:

$$\begin{aligned} \Sigma_b(y) &= c_b\left(\frac{y}{4} + O\left(\frac{\log y}{y}\right)\right)\left[\int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2 xdx\right] - c_b\Lambda Q(y)\int_1^y O(1)xdx \\ &= y\frac{\int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2}{\int \chi_{\frac{B_0}{4}}(\Lambda Q)^2} + O\left(\frac{1+y}{|\log b|}\right) \end{aligned} \tag{2.38}$$

and similarly for $y \leq 6B_0$:

$$\Lambda^i\Sigma_b(y) = y\frac{\int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2}{\int \chi_{\frac{B_0}{4}}(\Lambda Q)^2} + O\left(\frac{1+y}{|\log b|}\right) \quad \text{for} \quad 0 \leq i \leq 2.$$

The equation (2.34) and the cancellation $A\Lambda Q = H\Lambda Q = 0$ yield the bounds:

$$\int |H\Sigma_b|^2 \lesssim \frac{1}{|\log b|}, \quad \int \frac{1+|\log y|^2}{1+y^4}|H\Sigma_b|^2 \lesssim \frac{1}{|\log b|^2}, \tag{2.39}$$

$$\int \frac{1+|\log y|^2}{1+y^2}|AH\Sigma_b|^2 \lesssim b^2, \quad \int |H^2\Sigma_b|^2 \lesssim \frac{b^2}{|\log b|^2}. \tag{2.40}$$

Step 4 Construction of T_2.

Let

$$\Sigma_2 = \Sigma_b + T_1 - \Lambda T_1 - \frac{f''(Q)}{2y^2}T_1^2. \tag{2.41}$$

First observe from (1.4) that $f = gg'$ is odd and 2π periodic and thus:

$$\forall k \geq 0, \quad |f^{(2k)}(u)| + |f^{(2k)}(\pi - u)| \lesssim C_k|u|, \quad |f^{(2k+1)}(u)| \lesssim 1$$

which implies:

$$\forall k \geq 0, \quad |f^{(2k)}(Q)| \lesssim \frac{y}{1+y^2}. \tag{2.42}$$

We estimate from (2.12), (2.30), (2.31), (2.37): for $6B_0 \leq y \leq 2B_1$,

$$\Sigma_2(y) = O\left(\frac{|\log y|^2}{y}\right),$$

and for $y \leq 6B_0$, there holds the behavior (2.29) at the origin and the desired cancellation:

$$\begin{aligned}\Sigma_2(y) &= y\frac{\int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2}{\int \chi_{\frac{B_0}{4}}(\Lambda Q)^2} - y + O\left(\frac{1+y}{|\mathrm{log} b|}\right) + O\left(\frac{|\mathrm{log} y|^2}{1+y}\right) \\ &= O\left(\frac{1+y}{|\mathrm{log} b|}(1+|\log(y\sqrt{b})|\right).\end{aligned}$$

Remark 2.7. The above cancellation is due both to the presence of the T_1 term in the RHS of (2.41) which follows from the choice of modulation equation $b_s = -b^2$ to leading order and cancels the $y\log y$ growth of T_1, and the radiation term Σ_b which is designed to cancel the remaining y growth in $T_1 - \Lambda T_1$.

We similarly estimate: for $0 \leq i \leq 2$,

$$|\Lambda^i \Sigma_2| \lesssim \frac{y^3}{1+y^2}\left(\mathbf{1}_{y\leq 1} + \frac{1+\log(y\sqrt{b})}{|\mathrm{log} b|}\mathbf{1}_{1\leq y\leq 6B_0}\right) + \frac{(\mathrm{log} y)^2}{y}\mathbf{1}_{y\geq 6B_0}. \tag{2.43}$$

We now let T_2 be the solution to

$$HT_2 = \Sigma_2 \tag{2.44}$$

given by

$$T_2(y) = -\Gamma(y)\int_0^y \Sigma_2 \Lambda Q x dx + \Lambda Q(y)\int_0^y \Sigma_2 \Gamma x dx$$

which satisfies (2.29) and the estimate from (2.43):

$$\forall y \leq 2B_1, \quad |\Lambda^i T_2(y)| \lesssim \frac{y^5}{1+y^4}\left(\mathbf{1}_{y\leq 1} + \frac{1}{b|\mathrm{log} b|}\mathbf{1}_{y\geq 1}\right), \quad \text{for} \quad 0 \leq i \leq 3 \tag{2.45}$$

We also have the rougher bound:

$$\forall y \leq 2B_1, \quad |T_2(y)| \lesssim y^3. \tag{2.46}$$

T_2 also displays a small dependence on b through Σ_2 which will evaluated in the proof of Proposition 2.8.

Step 5 Construction of T_3.

Let

$$\Sigma_3 = 2T_2 - \Lambda T_2 - \frac{f''(Q)}{y^2}T_1 T_2 - \frac{1}{6}\frac{f^{(3)}(Q)}{y^2}T_1^3, \tag{2.47}$$

then Σ_3 satisfies (2.29) and we estimate from (2.31), (2.42), (2.45):

$$\forall y \leq 2B_1, \quad |\Lambda^i \Sigma_3(y)| \lesssim \frac{y^5}{1+y^4}\left(\mathbf{1}_{y\leq 1} + \frac{1}{b|\mathrm{log} b|}\mathbf{1}_{y\geq 1}\right), \quad \text{for} \quad 0 \leq i \leq 2 \tag{2.48}$$

We then let T_3 be the solution to

$$HT_3 = \Sigma_3 \tag{2.49}$$

given by :

$$T_3(y) = -\Gamma(y)\int_0^y \Sigma_3 \Lambda Q + \Lambda Q(y)\int_0^y \Sigma_3 \Gamma$$

which satisfies (2.29) and the estimates from (2.48):

$$\forall y \leq 2B_1, \quad |\Lambda^i T_3(y)| \lesssim \frac{y^7}{1+y^4}\left(\mathbf{1}_{y\leq 1} + \frac{1}{b|\mathrm{log} b|}\mathbf{1}_{y\geq 1}\right), \quad 0 \leq i \leq 1, \tag{2.50}$$

$$|T_3(y)| \lesssim y^3(1+y^2). \tag{2.51}$$

T_3 also displays a small dependence on b which will evaluated in the proof of Proposition 2.8. We claim the bounds for $i=0,1$:

$$\int_{y\leq 2B_1}|H\Lambda^iT_3|^2\lesssim\frac{|\log b|^2}{b^4},\quad \int_{y\leq 2B_1}\frac{1+|\log y|^2}{1+y^4}|H\Lambda^iT_3|^2\lesssim\frac{1}{b^4|\log b|^2} \tag{2.52}$$

$$\int\frac{1+|\log y|^2}{1+y^2}|AH\Lambda^iT_3|^2\lesssim\frac{|\log b|^6}{b^2},\quad \int_{y\leq 2B_1}|H^2\Lambda^iT_3|^2\lesssim\frac{1}{b^2|\log b|^2}. \tag{2.53}$$

Proof of (2.52), (2.53): Observe from a simple rescaling argument that for any function f:

$$H\Lambda f=2Hf+\Lambda Hf-\frac{\Lambda V}{y^2}f. \tag{2.54}$$

Hence from (2.49), (2.15):

$$H(\Lambda T_3)=2\Sigma_3+\Lambda\Sigma_3-\frac{\Lambda V}{y^2}T_3=2\Sigma_3+\Lambda\Sigma_3+O(y)$$

$$H^2(\Lambda T_3)=H(2\Sigma_3+\Lambda\Sigma_3)+O\left(\frac{1}{1+y}\right). \tag{2.55}$$

We thus estimate from (2.48), (1.20) for $i=0,1$:

$$\int_{y\leq 2B_1}|H\Lambda^iT_3|^2\lesssim\int_{y\leq 2B_1}\left|\frac{1+y}{b|\log b|}\right|^2\lesssim\frac{B_1^4}{b^2|\log b|^2}\lesssim\frac{|\log b|^2}{b^4}, \tag{2.56}$$

$$\int_{y\leq 2B_1}\frac{1+|\log y|^2}{1+y^4}|H\Lambda^iT_3|^2\lesssim\int_{y\leq 2B_1}\frac{1}{b^2|\log b|^2}\frac{(1+y^2)(1+|\log y|^2)}{1+y^4}\lesssim\frac{1}{b^4|\log b|^2},$$

and using the rough bound (2.51):

$$\int_{y\leq 2B_1}\frac{1+|\log y|^2}{1+y^2}|AH\Lambda^iT_3|^2\lesssim\int_{y\leq 2B_1}\frac{(1+y^4)(1+|\log y|^2)}{1+y^2}\lesssim B_1^4|\log b|^2\lesssim\frac{|\log b|^6}{b^2}.$$

The second bound in (2.53) is more subtle and requires further cancellations with respect to (2.50). Indeed, from (2.54), (2.47), (2.44):

$$H\Sigma_3=2HT_2-H\Lambda T_2+O\left(\frac{|\log y|^3}{y}\right)=\Lambda\Sigma_2+O\left(\frac{|\log y|^5}{y}\right),$$

$$H\Lambda\Sigma_3=-2\Lambda\Sigma_2-\Lambda^2\Sigma_2+O\left(\frac{|\log y|^5}{y}\right)$$

and injecting this into (2.55) with (2.43) yields[15]:

$$\begin{aligned}\int_{y\leq 2B_1}|H^2(\Lambda T_3)|^2 &\lesssim \int_{y\leq 2B_1}\left|\frac{y^3}{1+y^2}\left(\mathbf{1}_{y\leq 1}+\frac{1+\log(y\sqrt{b})}{|\log b|}\mathbf{1}_{1\leq y\leq 6B_0}\right)+\frac{(\log y)^2}{y}\mathbf{1}_{y\geq 6B_0}\right|^2\\ &\lesssim \frac{1}{b^2|\log b|^2}\end{aligned}$$

and (2.53) is proved.

step 6 Estimate on the error.

By construction, we have from (2.27):

$$\Psi_b=b^2\Sigma_b+b^4\Lambda T_3+\frac{1}{y^2}(R_1+R_2). \tag{2.57}$$

[15]the key here is that Σ_2 decays at infinity from the cancellation $H\Gamma=0$, and hence the control becomes independent of B_1

We inject into the formulas (2.25), (2.26) the rough bounds (2.31), (2.46), (2.51) and the definition of B_1 (1.20), and obtain the rough bound:

$$\forall y \leq 2B_1, \quad \left|\frac{d^i}{dy^i}\frac{R_1(y)}{y^2}\right| + \left|\frac{d^i}{dy^i}\frac{R_2(y)}{y^2}\right| \lesssim b^4 y^{5-i}\mathbf{1}_{y\leq 1} + \frac{b^4y^5|\mathrm{log} b|^C}{1+y^{2+i}}, \quad 0 \leq i \leq 4.$$

This yields:

$$\int_{y\leq 2B_1}\left|H\left(\frac{R_1}{y^2}\right)\right|^2 + \left|H\left(\frac{R_2}{y^2}\right)\right|^2 \lesssim b^8|\mathrm{log} b|^C\int_{y\leq 2B_1}\frac{y^{10}}{y^8} \lesssim b^6|\mathrm{log} b|^C,$$

$$\int_{y\leq 2B_1}\left|AH\left(\frac{R_1}{y^2}\right)\right|^2 + \left|AH\left(\frac{R_2}{y^2}\right)\right|^2 \lesssim b^8|\mathrm{log} b|^C\int_{y\leq 2B_1}\frac{y^{10}}{1+y^{10}} \lesssim b^7|\mathrm{log} b|^C,$$

$$\int_{y\leq 2B_1}\left|H^2\left(\frac{R_1}{y^2}\right)\right|^2 + \left|H^2\left(\frac{R_2}{y^2}\right)\right|^2 \lesssim b^8|\mathrm{log} b|^C\int_{y\leq 2B_1}\frac{y^{10}}{1+y^{12}} \lesssim b^8|\mathrm{log} b|^C.$$

Injecting these bounds together with (2.39), (2.40), (2.52), (2.53) into (2.57) yields (2.20), (2.21), (2.22), (2.23).
We now prove the flux computation (2.24). From (2.57), (2.34):

$$\begin{aligned}\frac{(H\Psi_b,\Phi_M)}{(\Lambda Q,\Phi_M)} &= \frac{1}{(\Lambda Q,\Phi_M)}\left[\left(-b^2c_b\chi_{\frac{B_0}{4}}\Lambda Q,\Phi_M\right) + O\left(C(M)b^3\right)\right]\\ &= -c_bb^2 + O\left(C(M)b^3\right) = -\frac{2b^2}{|\mathrm{log} b|} + O\left(\frac{b^2}{|\mathrm{log} b|^2}\right),\end{aligned}$$

for $|b| < b^*(M)$ small enough. This concludes the proof of Proposition 2.4

2.4. **Localization of the profile.** The Q_b profile constructed above displays a very specific growth in the parabolic zone $y \sim B_0 \sim \frac{1}{\sqrt{b}}$. We claim that further away, this growth becomes irrelevant in the description of the full solution, and we may therefore proceed to a simple localization procedure for $y \geq 2B_1 \sim \frac{|\mathrm{log} b|}{\sqrt{b}} \gg B_0$..

Proposition 2.8 (Localization of the profile)**.** *Let $(T_i)_{1\leq i\leq 3}$ be given by Proposition 2.4. Let a $\mathcal{C}^1$ map $s \mapsto b(s)$ defined on $[0,s_0]$ with a priori bound:$\forall s \in [0,s_0]$,*

$$0 < b(s) < b^*(M), \quad |b_s| \leq 10b^2. \tag{2.58}$$

Let the localized profile

$$\tilde{Q}_b(s,y) = Q + b\tilde{T}_1 + b^2\tilde{T}_2 + b^3\tilde{T}_3 = Q + \tilde{\alpha}$$

where

$$\tilde{T}_i = \chi_{B_1}T_i, \quad 1 \leq i \leq 3.$$

Then

$$\partial_s\tilde{Q}_b - \Delta\tilde{Q}_b - \frac{\lambda_s}{\lambda}\Lambda\tilde{Q}_b + \frac{f(\tilde{Q}_b)}{y^2} = Mod(t) + \tilde{\Psi}_b \tag{2.59}$$

with

$$Mod(t) = -\left(\frac{\lambda_s}{\lambda} + b\right)\Lambda\tilde{Q}_b + (b_s + b^2)(\tilde{T}_1 + 2b\tilde{T}_2) \tag{2.60}$$

and where $\tilde{\Psi}_b$ satisfies the bounds on $[0,s_0]$:
(i) Weighted bounds:

$$\int |H\tilde{\Psi}_b|^2 \lesssim b^4|\mathrm{log} b|^2, \tag{2.61}$$

$$\int \frac{1+|\mathrm{log} y|^2}{1+y^4}|H\tilde{\Psi}_b|^2 \lesssim \frac{b^4}{|\mathrm{log} b|^2}, \tag{2.62}$$

$$\int \frac{1+|\log y|^2}{1+y^2}|AH\tilde{\Psi}_b|^2 \lesssim \frac{b^5}{|\log b|^2}, \quad (2.63)$$

$$\int |H^2\tilde{\Psi}_b|^2 \lesssim \frac{b^6}{|\log b|^2}. \quad (2.64)$$

(ii) Flux computation: Let Φ_M *be given by* (3.5)*, then:*

$$\frac{(H\tilde{\Psi}_b,\Phi_M)}{(\Lambda Q,\Phi_M)} = -\frac{2b^2}{|\log b|} + O\left(\frac{b^2}{|\log b|^2}\right), \quad (2.65)$$

Proof of Proposition 2.8

step 1 Localization.

Let

$$\Psi_b^{(1)} = -b^2(\tilde{T}_1+2b\tilde{T}_2) - \Delta\tilde{Q}_b + b\Lambda\tilde{Q}_b + \frac{f(\tilde{Q}_b)}{y^2}.$$

We compute the action of localization which produces an error localized in $[B_1, 2B_1]$ up to the term $(1-\chi_{B_1})\Lambda Q$:

$$\begin{aligned}\Psi_b^{(1)} &= \chi_{B_1}\Psi_b + b(1-\chi_{B_1})\Lambda Q + b\Lambda\chi_{B_1}\alpha - \alpha\Delta\chi_{B_1} - 2\partial_y\chi_{B_1}\partial_y\alpha \\ &+ \frac{1}{y^2}\left[f(Q+\chi_{B_1}\alpha) - f(Q) - \chi_{B_1}(f(Q+\alpha)-f(Q))\right]. \end{aligned} \quad (2.66)$$

We estimate in brute force from (2.31), (2.46), (2.51) and the choice of B_1:

$$\forall y \le 2B_1, \quad |\alpha(y)| \lesssim by\left(|\log y| + \frac{by^2}{|\log b|}\right) \lesssim by|\log y|$$

and thus:

$$|b(1-\chi_{B_1})\Lambda Q + b\Lambda\chi_{B_1}\alpha - \alpha\Delta\chi_{B_1} - 2\partial_y\chi_{B_1}\partial_y\alpha| \lesssim \frac{b}{y}\mathbf{1}_{y\ge B_1} + b^2 y\log y\mathbf{1}_{B_1\le y\le 2B_1},$$

$$\begin{aligned}\left|\frac{1}{y^2}\left[f(Q+\chi_{B_1}\alpha) - f(Q) - \chi_{B_1}(f(Q+\alpha)-f(Q))\right]\right| &\lesssim \frac{|\alpha(y)|}{y^2}|\mathbf{1}_{B_1\le y\le 2B_1} \\ &\lesssim \frac{b\log y}{y}\mathbf{1}_{B_1\le y\le 2B_1}\end{aligned}$$

from which using (2.20), (2.21), (2.22), (2.23):

$$\int |H\Psi_b^{(1)}|^2 \lesssim b^4|\log b|^2 + \int_{B_1\le y\le 2B_1}\left[\frac{b^2|\log y|^2}{y^6} + \frac{b^4|\log y|^2}{y^2}\right] \lesssim b^4|\log b|^2,$$

$$\int \frac{1+|\log y|^2}{1+y^4}|H\Psi_b^{(1)}|^2 \lesssim \frac{b^4}{|\log b|^2} + \int_{B_1\le y\le 2B_1}\left[\frac{b^2|\log y|^2}{y^8} + \frac{b^4|\log y|^2}{y^4}\right] \lesssim \frac{b^4}{|\log b|^2},$$

$$\int \frac{1+|\log y|^2}{1+y^2}|AH\Psi_b^{(1)}|^2 \lesssim \frac{b^5}{|\log b|^2} + \int_{B_1\le y\le 2B_1}\frac{1+|\log y|^2}{1+y^2}\left[\frac{b^2}{y^8} + \frac{b^4|\log y|^2}{y^4}\right] \lesssim \frac{b^5}{|\log b|^2},$$

$$\begin{aligned}\int |H^2\Psi_b^{(1)}|^2 &\lesssim \frac{b^6}{|\log b|^2} + \int_{B_1\le y\le 2B_1}\left[\frac{b^2|\log y|^2}{y^{10}} + \frac{b^4|\log y|^2}{y^6}\right] \\ &\lesssim \frac{b^6}{|\log b|^2} + \frac{b^2|\log b|^2}{B_1^8} + \frac{b^4|\log b|^2}{B_1^4} \lesssim \frac{b^6}{|\log b|^2}.\end{aligned}$$

Remark 2.9. This last estimate and (2.56) govern the choice $B_1 = \frac{|\log b|}{\sqrt{b}}$.

step 2 Control of time derivatives.

We now compute from (2.59):

$$\tilde{\Psi}_b=\Psi_b^{(1)}+\tilde{R},\quad \tilde{R}=b_s(3b^2\tilde{T}_3+b\frac{\partial\tilde{T}_1}{\partial b}+b^2\frac{\partial\tilde{T}_2}{\partial b}+b^3\frac{\partial\tilde{T}_3}{\partial b}) \tag{2.67}$$

and estimate all terms. From (2.34),

$$\frac{\partial c_b}{\partial b}=O\left(\frac{1}{b|\log b|^2}\right),\quad \frac{\partial d_b}{\partial b}=O\left(\frac{1}{b^2|\log b|}\right) \tag{2.68}$$

and thus from (2.35):

$$\begin{aligned}\frac{\partial\Sigma_b}{\partial b}(y) &= \frac{\partial c_b}{\partial b}T_1\mathbf{1}_{y\leq\frac{B_0}{2}}+O\left(\frac{1}{b^2y\log b|}\mathbf{1}_{\frac{B_0}{2}\leq y\leq 6B_0}\right)\\ &= O\left(\frac{y}{b|\log b|}\mathbf{1}_{y\leq B_0}+\frac{1}{b^2y|\log b|}\mathbf{1}_{\frac{B_0}{2}\leq y\leq 6B_0}\right).\end{aligned}$$

We inject this estimate into the explicit formulas for T_2,T_3 and conclude:

$$\frac{\partial\tilde{T}_1}{\partial b}=O\left(\frac{y\log y}{b}\mathbf{1}_{\frac{B_1}{2}\leq y\leq 2B_1}\right), \tag{2.69}$$

$$\frac{\partial\tilde{T}_2}{\partial b}=O\left(\frac{1+y^3}{b|\log b|}\mathbf{1}_{y\leq B_0}+\frac{y}{b^2|\log b|}\mathbf{1}_{\frac{B_0}{2}\leq y\leq 2B_1}\right), \tag{2.70}$$

$$\frac{\partial\tilde{T}_3}{\partial b}=O\left(\frac{1+y^5}{b|\log b|}\mathbf{1}_{y\leq B_0}+\frac{y^3}{b^2|\log b|}\mathbf{1}_{\frac{B_0}{2}\leq y\leq 2B_1}\right). \tag{2.71}$$

This yields together with the a priori bound (2.58) the pointwise control:

$$|\tilde{R}|\lesssim b^2\left[\frac{b(1+y^3)}{|\log b|}\mathbf{1}_{y\leq B_0}+\frac{by^3}{|\log b|}\mathbf{1}_{\frac{B_0}{2}\leq y\leq 6B_1}+y\log y\mathbf{1}_{\frac{B_1}{2}\leq y\leq 2B_1}\right]$$

and hence the bounds:

$$\begin{aligned}\int|H\tilde{R}|^2 &\lesssim b^4\int\left|\frac{b(1+y)}{|\log b|}\mathbf{1}_{y\leq B_0}+\frac{by}{|\log b|}\mathbf{1}_{\frac{B_0}{2}\leq y\leq 6B_1}+\frac{\log y}{y}\mathbf{1}_{\frac{B_1}{2}\leq y\leq 2B_1}\right|^2\\ &\lesssim b^4|\log b|^2,\end{aligned}$$

$$\begin{aligned}&\int\frac{1+|\log y|^2}{1+y^4}|H\tilde{R}|^2+\int\frac{1+|\log y|^2}{1+y^2}|AH\tilde{R}|^2\\ \lesssim\ & b^4\int\frac{1+|\log y|^2}{1+y^4}\left|\frac{b(1+y)}{|\log b|}\mathbf{1}_{y\leq B_0}+\frac{by}{|\log b|}\mathbf{1}_{\frac{B_0}{2}\leq y\leq 6B_1}+\frac{\log y}{y}\mathbf{1}_{\frac{B_1}{2}\leq y\leq 2B_1}\right|^2\\ \lesssim\ & \frac{b^5}{|\log b|^2}\end{aligned}$$

We now track for more cancellation when applying H^2. Indeed, from (2.52), (2.50):

$$\int|H^2\tilde{T}_3|^2\lesssim\frac{1}{b^2|\log b|^2}+\int\left|\frac{y^3}{by^4|\log b|}\mathbf{1}_{B_1\leq y\leq 2B_1}\right|^2\lesssim\frac{1}{b^2|\log b|^2}.$$

From direct inspection:

$$\int\left|H^2\left(\frac{\partial T_1}{\partial b}\right)\right|^2\lesssim\int_{B_1\leq y\leq 2B_1}\left|\frac{\log y}{by^3}\right|^2\lesssim\frac{|\log b|^2}{b^2B_1^4}\lesssim\frac{1}{|\log b|^2}.$$

Next:

$$H^2\left(\frac{\partial\tilde{T}_2}{\partial b}\right)=\chi_{B_1}H^2\left(\frac{\partial T_2}{\partial b}\right)+O\left(\frac{1}{y^3b^2|\log b|}\mathbf{1}_{\frac{B_1}{2}\leq y\leq 2B_1}\right)$$

and from (2.44), (2.41):

$$\begin{aligned}H^2\left(\frac{\partial T_2}{\partial b}\right) &= H\left(\frac{\partial\Sigma_2}{\partial b}\right)=H\left(\frac{\partial\Sigma}{\partial b}\right)\\ &= O\left(\frac{1}{b(1+y)|\log b|^2}\mathbf{1}_{y\leq 2B_0}+\frac{1}{b(1+y)|\log b|}\mathbf{1}_{\frac{B_0}{2}\leq y\leq 2B_0}\right)\end{aligned}$$

which yields the bound:

$$\begin{aligned}\int\left|H^2\left(\frac{\partial\tilde{T}_2}{\partial b}\right)\right|^2 &\lesssim \int\left|\frac{1}{b(1+y)|\log b|^2}\mathbf{1}_{y\leq 2B_0}+\frac{1}{b(1+y)|\log b|}\mathbf{1}_{\frac{B_0}{2}\leq y\leq 2B_0}\right.\\ &+ \left.\frac{1}{y^3b^2|\log b|}\mathbf{1}_{\frac{B_1}{2}\leq y\leq 2B_1}\right|^2\lesssim\frac{1}{b^2|\log b|^2}.\end{aligned}$$

Similarly,

$$\begin{aligned}H^2\left(\frac{\partial\tilde{T}_3}{\partial b}\right) &= \chi_{B_1}H^2\left(\frac{\partial T_3}{\partial b}\right)+O\left(\frac{1}{yb^2|\log b|}\mathbf{1}_{\frac{B_1}{2}\leq y\leq 2B_1}\right)\\ &= O\left(\frac{1+y}{b|\log b|}\mathbf{1}_{y\leq B_0}+\frac{1}{yb^2|\log b|}\mathbf{1}_{\frac{B_0}{2}\leq y\leq 2B_0}+\frac{1}{yb^2|\log b|}\mathbf{1}_{\frac{B_1}{2}\leq y\leq 2B_1}\right)\end{aligned}$$

from which:

$$\begin{aligned}\int\left|H^2\left(\frac{\partial\tilde{T}_3}{\partial b}\right)\right|^2 &\lesssim \int\left|\frac{1+y}{b|\log b|}\mathbf{1}_{y\leq B_0}+\frac{1}{yb^2|\log b|}\mathbf{1}_{\frac{B_0}{2}\leq y\leq 2B_0}+\frac{1}{yb^2|\log b|}\mathbf{1}_{\frac{B_1}{2}\leq y\leq 2B_1}\right|^2\\ &\lesssim \frac{1}{b^4|\log b|^2}.\end{aligned}$$

We inject these estimates into (2.67) and obtain:

$$\int|H^2\tilde{R}|^2\lesssim\frac{b^6}{|\log b|^2}.$$

Injecting the collection of estimates of step 1 and step 2 into (2.67) now yields the expected bounds (2.61), (2.62), (2.63), (2.64).

step 3 Flux computation.

By definition, $\Psi_b^{(1)}=\Psi_b$ on $\mathrm{Supp}(\Phi_M)\subset[0,2M]$, and from (2.69), (2.70), (2.71) and (2.50):

$$|\tilde{R}|\lesssim C(M)|b_s|b\lesssim C(M)b^3\ \ \text{on } \mathrm{Supp}(\Phi_M).$$

This estimate together with (2.24) now yields (2.65).
This concludes the proof of Proposition 2.8.

We introduce a second localization of the profile near B_0 which will be used only to capture some further cancellation in the proof of the regularity claim (1.14).

Lemma 2.10 (Second localization)**.** *Let a $\mathcal{C}^1$ map $s\mapsto b(s)$ defined on $[0,s_0]$ with a priori bound* (1.14)*. Let the localized profile*

$$\hat{Q}_b(s,y)=Q+b\hat{T}_1+b^2\hat{T}_2+b^3\hat{T}_3=Q+\hat{\alpha}\qquad(2.72)$$

where

$$\hat{T}_i = \chi_{B_0} T_i, \quad 1 \leq i \leq 3.$$

Let the radiation:

$$\zeta_b = \tilde{\alpha} - \hat{\alpha} \tag{2.73}$$

and the error

$$\partial_s \hat{Q}_b - \Delta \hat{Q}_b - \frac{\lambda_s}{\lambda} \Lambda \hat{Q}_b + \frac{f(\hat{Q}_b)}{y^2} = \widehat{Mod}(t) + \hat{\Psi}_b$$

with

$$\widehat{Mod}(t) = -\left(\frac{\lambda_s}{\lambda} + b\right) \Lambda \hat{Q}_b + (b_s + b^2)(\hat{T}_1 + 2b\hat{T}_2). \tag{2.74}$$

Then there holds the bounds:

$$\int |H\zeta_b|^2 \lesssim b^2 |\mathrm{log} b|^2, \quad \Sigma_{i=0}^2 \int \frac{|\partial_y^i \zeta_b|^2}{1 + y^{8-2i}} \lesssim b^4 |\mathrm{log} b|^C, \tag{2.75}$$

$$\Sigma_{i=0}^2 \int \frac{|\partial_y^i \zeta_b|^2}{1 + y^{2(3-i)}} \lesssim b^3 |\mathrm{log} b|^C, \tag{2.76}$$

$$Supp(H\hat{\Psi}_b) \subset [0, 2B_0] \quad \textit{and} \quad \int |H\hat{\Psi}_b|^2 \lesssim b^4 |\mathrm{log} b|^2. \tag{2.77}$$

Remark 2.11. Note that this localization near B_0 displays the same properties like the one near B_0 at the $\mathcal{H}^2$ level and (2.61), (2.77) are comparable. The estimate corresponding to (2.64) would however be worse for $\hat{\Psi}_b$ due to the terms induced by localization, see Remark 2.9. Hence we will use the B_1 localization to control high derivatives norms, see Proposition 3.4 and the control of $\mathcal{E}_4$ below, and B_0 localization for lower order control, see section 4.1 and the control of $\mathcal{E}_2$.

Proof of Lemma 2.10 By construction,

$$\zeta_b = (\chi_{B_1} - \chi_{B_0})(bT_1 + b^2 T_2 + b^3 T_3)$$

and thus from (2.45), (2.50):

$$\int |H\zeta_b|^2 \lesssim \int_{y \leq 2B_1} \left| \frac{by \mathrm{log} y}{y^2} + \frac{b^2 y}{by^2 |\mathrm{log} b|} + \frac{b^3 y^3}{by^2 |\mathrm{log} b|} \right|^2 \lesssim b^2 |\mathrm{log} b|^2,$$

$$\Sigma_{i=0}^2 \int \frac{|\partial_y^i \zeta_b|^2}{1 + y^{8-2i}} \lesssim \int_{B_0 \leq y \leq 2B_1} \frac{1}{1 + y^8} \left[b^2 y^2 |\mathrm{log} y|^2 + \frac{b^6 y^6}{b^2 |\mathrm{log} b|^2} \right] \lesssim b^4 |\mathrm{log} b|^C,$$

$$\Sigma_{i=0}^2 \int \frac{|\partial_y^i \zeta_b|^2}{1 + y^{2(3-i)}} \lesssim \int_{B_0 \leq y \leq 2B_1} \frac{1}{1 + y^6} \left[b^2 y^2 |\mathrm{log} y|^2 + \frac{b^6 y^6}{b^2 |\mathrm{log} b|^2} \right] \lesssim b^3 |\mathrm{log} b|^C.$$

The localization property (2.77) directly follows from the analogue of the formula (2.66), (2.67) for $\hat{\Psi}_b$ and the cancellation $H\Lambda Q = 0$, while the proof of the estimate (2.77) is very similar to the one of (2.61) and left to the reader.

This concludes the proof of Lemma 2.10.

3. **The trapped regime**

This section is devoted to the description of the set of initial data and the corresponding trapped regime in which the singularity formation described by Theorem 1.1 will occur.

3.1. **Setting the bootstrap.** We describe in this section the set of initial data leading to the blow up scenario of Theorem 1.1. Let a 1-corotational map

$$v_0 \in \dot{H}^1 \cap \dot{H}^4 \quad \text{with} \quad \|\nabla(v_0 - \mathcal{Q})\|_{L^2} \ll 1$$

and $v(t) \in \mathcal{C}([0,T), \dot{H}^1 \cap \dot{H}^4)$, $0 < T \leq +\infty$ be the corresponding solution to (1.1). First recall from standard argument the blow up criterion:

$$T < +\infty \quad \text{implies} \quad \|\Delta v(t)\|_{L^2} \to +\infty \quad \text{as} \quad t \to T. \tag{3.1}$$

From Lemma A.1, v admits on a small time interval $[0, t_1]$ a decomposition

$$v(t,x) = \left| \begin{array}{l} g(u(t,r))\cos\theta \\ g(u(t,r))\sin\theta \\ z(u(t,r)) \end{array} \right. \tag{3.2}$$

where $\tilde{\varepsilon}(t,r) = u(t,r) - Q(r)$ satisfies the boundary condition (A.3) and the regularity (A.4), (A.5) displayed in Lemma A.1. Moreover, from the initial smallness

$$\|\nabla \tilde{\varepsilon}(0)\|_{L^2} + \|\frac{\tilde{\varepsilon}(0)}{y}\|_{L^2} \ll 1,$$

we may from standard modulation argument[16] introduce for each fixed time $t \in [0, t_1]$ the unique decomposition

$$u(t,r) = (\tilde{Q}_{b(t)} + \varepsilon)\left(t, \frac{y}{\lambda(t)}\right), \quad \|\nabla\varepsilon(t)\|_{L^2} + \|\frac{\varepsilon(t)}{y}\|_{L^2} + |b(t)| \ll 1, \quad \lambda(t) > 0 \tag{3.3}$$

where $\varepsilon(t)$ satisfies the orthogonality conditions:

$$\forall t \in [0, t_1], \quad (\varepsilon(t), \Phi_M) = (\varepsilon(t), H\Phi_M) = 0. \tag{3.4}$$

Here given $M > 0$ large enough, Φ_M corresponds to the fixed direction

$$\Phi_M = \chi_M \Lambda Q - c_M H(\chi_M \Lambda Q) \tag{3.5}$$

with

$$c_M = \frac{(\chi_M \Lambda Q, T_1)}{(H(\chi_M \Lambda Q), T_1)} = c_\chi \frac{M^2}{4}(1 + o_{M \to +\infty}(1)).$$

Observe by construction that

$$\int |\Phi_M|^2 \lesssim |\text{log}M|, \quad (\Phi_M, T_1) = 0 \tag{3.6}$$

and the scalar products

$$(\Lambda Q, \Phi_M) = (-HT_1, \Phi_M) = (\chi_M \Lambda Q, \Lambda Q) = 4\text{log}M(1 + o_{M \to +\infty}(1)) \tag{3.7}$$

are non degenerate. The existence of the decomposition (3.3) is then a standard consequence of the implicit function theorem and the explicit relations

$$\left(\frac{\partial}{\partial\lambda}(\tilde{Q}_b)_\lambda, \ \frac{\partial}{\partial b}(\tilde{Q}_b)_\lambda\right)|_{\lambda=1, b=0} = (\Lambda Q, T_1)$$

which ensure the nondegeneracy of the Jacobian:

$$\begin{aligned} \left| \begin{array}{ll} (\frac{\partial}{\partial\lambda}(\tilde{Q}_b)_\lambda, \Phi_M) & (\frac{\partial}{\partial b}(\tilde{Q}_b)_\lambda, \Phi_M) \\ (\frac{\partial}{\partial\lambda}(\tilde{Q}_b)_\lambda, H\Phi_M) & (\frac{\partial}{\partial b}(\tilde{Q}_b)_\lambda, H\Phi_M) \end{array} \right|_{\lambda=1, b=0} &= \left| \begin{array}{ll} (\Lambda Q, \Phi_M) & 0 \\ 0 & (T_1, H\Phi_M) \end{array} \right| \\ &= -(\Lambda Q, \Phi_M)^2 \neq 0. \end{aligned}$$

[16] see for example [13], [15], [22] for a further introduction to modulation.

From Lemma A.1, we may measure the regularity of the map through the following norms of ε: the energy norm

$$\|\varepsilon\|_{\mathcal{H}}^2 = \int |\partial_y \varepsilon|^2 + \int \frac{|\varepsilon|^2}{y^2}, \tag{3.8}$$

and higher order Sobolev norms adapted to the linearized operator

$$\mathcal{E}_{2k} = \int |H^k \varepsilon|^2, \quad 1 \leq k \leq 2. \tag{3.9}$$

We now assume the following bounds on initial data which describe an open $\dot{H}^1 \cap H^4$ affine space of 1-corotational initial data around $\mathcal{Q}$:

- Smallness and positivity of b(0) :

$$0 < b(0) < b^*(M) \ll 1. \tag{3.10}$$

- Smallness of the excess of energy:

$$\|\varepsilon(0)\|_{\mathcal{H}}^2 \leq b^2(0), \tag{3.11}$$

$$|\mathcal{E}_2(0)| + |\mathcal{E}_4(0)| \leq [b(0)]^{10}. \tag{3.12}$$

The regularity of the full map $v \in \mathcal{C}^0([0,T), \dot{H}^1 \cap \dot{H}^4)$ by the parabolic heat flow and Lemma A.1 ensure that these estimates hold on some small enough time interval $[0,t_1]$, and from standard argument[17], there also holds the regularity $(\lambda, b) \in \mathcal{C}^1([0,t_1], \mathbb{R}_+^* \times \mathbb{R})$. Given a large enough universal constant $K > 0$ -independent of M-, we assume on $[0,t_1]$ the following bootstrap bounds:

- Control of b(t) :

$$0 < b(t) < 10b(0). \tag{3.13}$$

- Control of the radiation:

$$\|\varepsilon(t)\|_{\mathcal{H}}^2 \leq 10\sqrt{b(0)}, \tag{3.14}$$

$$|\mathcal{E}_2(t)| \leq K b^2(t) |\mathrm{log} b(t)|^5, \tag{3.15}$$

$$|\mathcal{E}_4(t)| \leq K \frac{b^4(t)}{|\mathrm{log} b(t)|^2}. \tag{3.16}$$

The following proposition describes the contraction of the bootstrap regime and is the core of the proof.

Proposition 3.1 (Bootstrap control of b and ε)**.** *Assume that K in* (3.13), (3.14), (3.15) *and* (3.16) *has been chosen large enough. Then,* $\forall t \in [0,t_1]$ *:*

$$0 < b(t) < 2b(0), \tag{3.17}$$

$$\|\varepsilon(t)\|_{\mathcal{H}}^2 \leq \sqrt{b(0)}, \tag{3.18}$$

$$|\mathcal{E}_2(t)| \leq \frac{K}{2} b^2(t) |\mathrm{log} b(t)|^5, \tag{3.19}$$

$$|\mathcal{E}_4(t)| \leq \frac{K}{2} \frac{b^4(t)}{|\mathrm{log} b(t)|^2}. \tag{3.20}$$

In other words, the solution is trapped in the regime (3.17), (3.18), (3.19), (3.20) and thus $t_1 = T$, the life time of the solution. We will in section 4.2 that this easily implies finite time blow up $T < +\infty$ in the regime described by Theorem 1.1. The rest of this section is devoted to developing the tools needed for the proof of Proposition 3.1 which will be completed in section 4.1.

[17] see [13] for more details in a completly similar setting.

3.2. **Equation for the radiation.** Let us now pass to renormalized variables[18]:

$$u(t,r)=z(s,y), \quad y=\frac{r}{\lambda(t)}, \quad s(t)=\int_0^t \frac{d\tau}{\lambda^2(\tau)} \tag{3.21}$$

and recall from (3.3) the decomposition of the flow:

$$z(s,y)=(Q+\tilde{\alpha}+\varepsilon)(s,y).$$

We will use the rescaling formula:

$$\partial_t u=\frac{1}{\lambda^2(t)}(\partial_s z-\frac{\lambda_s}{\lambda}\Lambda z)_\lambda$$

to derive the equation for ε in renormalized variables:

$$\partial_s\varepsilon-\frac{\lambda_s}{\lambda}\Lambda\varepsilon+H\varepsilon=F-Mod=\mathcal{F}. \tag{3.22}$$

Here H is the linearized operator given by (2.9), Mod(t) is given by (2.60),

$$F=-\tilde{\Psi}_b+L(\varepsilon)-N(\varepsilon) \tag{3.23}$$

where L is the linear operator corresponding to the error in the linearized operator from Q to $\tilde{Q}_b$:

$$L(\varepsilon)=\frac{f'(Q)-f'(\tilde{Q}_b)}{y^2}\varepsilon \tag{3.24}$$

and the remainder term is the purely nonlinear term:

$$N(\varepsilon)=\frac{f(\tilde{Q}_b+\varepsilon)-f(\tilde{Q}_b)-\varepsilon f'(\tilde{Q}_b)}{y^2}. \tag{3.25}$$

We also need to write the flow (3.22) in original variables (t,r). Let

$$w(t,r)=\varepsilon(s,y)$$

and the rescaled operators be:

$$A_\lambda=-\partial_r+\frac{Z_\lambda}{r}, \quad A^*_\lambda=\partial_r+\frac{1+Z_\lambda}{r},$$

$$H_\lambda=A^*_\lambda A_\lambda=-\Delta+\frac{V_\lambda}{r^2}, \quad \tilde{H}_\lambda=A_\lambda A^*_\lambda=-\Delta+\frac{\tilde{V}_\lambda}{r^2}, \tag{3.26}$$

then (3.22) becomes:

$$\partial_t w+H_\lambda w=\frac{1}{\lambda^2}\mathcal{F}_\lambda \tag{3.27}$$

3.3. **Modulation equations.** Let us now compute the modulation equations for (b,λ) as a consequence of the choice of orthogonality conditions (3.4).

Lemma 3.2 (Modulation equations)**.** *There holds the bound on the modulation parameters :*

$$\left|\frac{\lambda_s}{\lambda}+b\right|\lesssim\frac{b^2}{|\text{log}b|}+\frac{1}{\sqrt{\text{log}M}}\sqrt{\mathcal{E}_4}, \tag{3.28}$$

$$\left|b_s+b^2\left(1+\frac{2}{|\text{log}b|}\right)\right|\lesssim\frac{1}{\sqrt{\text{log}M}}\left(\sqrt{\mathcal{E}_4}+\frac{b^2}{|\text{log}b|}\right). \tag{3.29}$$

Remark 3.3. Note that this implies in the bootstrap the rough bounds:

$$|b_s|+\left|\frac{\lambda_s}{\lambda}+b\right|\leq 2b^2. \tag{3.30}$$

and in particular (2.58) holds.

[18] we will show that the renormalized time is global: $\lim_{t\to T}s(t)\to=+\infty$, see (4.16).

Proof of Lemma 3.2

step 1 Law for b.

Let

$$\mathcal{V}(t) = |b_s + b^2| + \left|\frac{\lambda_s}{\lambda} + b\right|$$

We take the inner product of (3.22) with $H\Phi_M$ and compute using (3.4):

$$\begin{aligned}(\mathrm{Mod}(t), H\Phi_M) &= -(\tilde{\Psi}_b, H\Phi_M) - (H\varepsilon, H\Phi_M) \\ &- \left(-\frac{\lambda_s}{\lambda}\Lambda\varepsilon - L(\varepsilon) + N(\varepsilon), H\Phi_M\right).\end{aligned} \tag{3.31}$$

We first compute from the construction of the profile, (2.60), the relations $H\Lambda Q = 0$, $HT_1 = -\Lambda Q$, and the localization $\mathrm{Supp}(\Phi_M) \subset [0, 2M]$ from (3.5):

$$\begin{aligned}(H\left(Mod(t)\right), \Phi_M) &= -\left(b + \frac{\lambda_s}{\lambda}\right)\left(H\Lambda\tilde{Q}_b, \Phi_M\right) + \left(b_s + b^2\right)\left(H\left(\tilde{T}_1 + 2b\tilde{T}_2\right), \Phi_M\right) \\ &= -(\Lambda Q, \Phi_M)(b_s + b^2) + O\left(c(M)b|\mathcal{V}(t)|\right).\end{aligned}$$

The linear term in (3.31) is estimated[19] from (3.6):

$$|(H\varepsilon, H\Phi_M)| \lesssim \|H^2\varepsilon\|_{L^2}\sqrt{\log M} = \sqrt{\log M \mathcal{E}_4}$$

and the remaining nonlinear term is estimated using the Hardy bounds of Appendix B:

$$\left|\left(-\frac{\lambda_s}{\lambda}\Lambda\varepsilon + L(\varepsilon) + N(\varepsilon), H\Phi_M\right)\right| \lesssim C(M)b(\sqrt{\mathcal{E}_4} + |\mathcal{V}(t)|).$$

We inject these estimates into (3.31) and conclude from (3.7) and the fundamental flux computation (2.65):

$$b_s + b^2 = \frac{(\tilde{\Psi}_b, H\Phi_M)}{(\Lambda Q, \Phi_M)} + O\left(\frac{\sqrt{\log M \mathcal{E}_4}}{\log M}\right) + O\left(C(M)b|\mathcal{V}(t)\right)$$

and hence the first modulation equation:

$$b_s + b^2 = -\frac{2b^2}{|\log b|}\left(1 + O\left(\frac{1}{|\log b|}\right)\right) + O\left(\sqrt{\frac{\mathcal{E}_4}{\log M}} + C(M)b|\mathcal{V}(t)|\right). \tag{3.32}$$

step 2 Degeneracy of the law for λ.

We now take the inner product of (3.22) with Φ_M and obtain:

$$(\mathrm{Mod}(t), \Phi_M) = -(\tilde{\Psi}_b, \Phi_M) - (H\varepsilon, \Phi_M) - \left(-\frac{\lambda_s}{\lambda}\Lambda\varepsilon + L(\varepsilon) + N(\varepsilon), \Phi_M\right).$$

Note first that the choice of orthogonality conditions (3.4) gets rid of the linear term in ε:

$$(H\varepsilon, \Phi_M) = 0.$$

[19] Observe that we do not use the interpolated bounds of Lemma B.3 but directly the definition (3.9) of $\mathcal{E}_4$, and hence the dependence of the constant in M is explicit what is crucial for the analysis.

Next, we compute from (3.7) and the orthogonality (3.6):

$$\begin{aligned}(\mathrm{Mod}(t), \Phi_M) &= -\left(b + \frac{\lambda_s}{\lambda}\right)\left(\Lambda\tilde{Q}_b, \Phi_M\right) + \left(b_s + b^2\right)\left(\tilde{T}_1 + 2b\tilde{T}_2, \Phi_M\right)\\ &= -4\mathrm{log}M(1 + o_{M\to+\infty}(1))\left(\frac{\lambda_s}{\lambda} + b\right) + O\left(C(M)b|\mathcal{V}(t)|\right).\end{aligned}$$

and observe the cancellation from (2.36), (3.6):

$$\left|\left(\tilde{\Psi}_b, \Phi_M\right)\right| \lesssim b^2|(\Sigma_b, \Phi_M)| + O(C(M)b^3) = c_b b^2|(T_1, \Phi_M)| + O(C(M)b^3) = O(C(M)b^3).$$

Nonlinear terms are easily estimated using the Hardy bounds of Appendix B:

$$\left|\left(-\frac{\lambda_s}{\lambda}\Lambda\varepsilon + L(\varepsilon) + N(\varepsilon), \Phi_M\right)\right| \lesssim C(M)b\left(\sqrt{\mathcal{E}_4} + |\mathcal{V}(t)|\right).$$

We thus obtain the modulation equation for scaling:

$$\left|\frac{\lambda_s}{\lambda} + b\right| \lesssim b^3C(M) + bC(M)O\left(\sqrt{\mathcal{E}_4} + |\mathcal{V}(t)|\right).$$

Combining this with (3.32) yields the bound

$$|\mathcal{V}(t)| \lesssim \frac{b^2}{|\mathrm{log}b|} + \frac{1}{\sqrt{logM}}\sqrt{\mathcal{E}_4}$$

which together with (3.32) again now implies the refined bound (3.29). This concludes the proof of Lemma 3.2.

3.4. **The Lyapounov monotonicity.** We now turn to the core of the argument which is the derivation of a suitable Lyapounov functional at the Sobolev $\dot{H}^4$ level. The parabolic structure will yield further dissipation with respect to the analysis of dispersive problems in [22], [19], what will allow us to treat a general metric g.

Proposition 3.4 (Lyapounov monotonicity)**.** *There holds:*

$$\frac{d}{dt}\left\{\frac{1}{\lambda^6}\left[\mathcal{E}_4 + O\left(\sqrt{b}\frac{b^4}{|\mathrm{log}b|^2}\right)\right]\right\} \leq C\frac{b}{\lambda^8}\left[\frac{\mathcal{E}_4}{\sqrt{\mathrm{log}M}} + \frac{b^4}{|\mathrm{log}b|^2} + \frac{b^2}{|\mathrm{log}b|}\sqrt{\mathcal{E}_4}\right] \quad (3.33)$$

for some universal constant $C > 0$ *independent of* M *and of the bootstrap constant* K *in* (3.13)*,* (3.14)*,* (3.15)*,* (3.16)*, provided* $b^*(M)$ *in* (3.10) *has been chosen small enough.*

Proof of Proposition 3.4

The proof relies on the derivation of the energy identity for suitable derivatives of ε seen in original variables ie w, and repulsivity properties of the corresponding time dependent Hamiltonian $\tilde{H}_\lambda$. The control of the solution is then ensured thanks to coercivity properties of the iterated Hamiltonian H, H^2 under the orthogonality conditions (3.4), see Lemma B.2. Nonlinear terms will be estimated using the interpolated bounds of Lemma B.3 which will be implicitly used all along the proof.

step 1 Suitable derivatives.

We define the derivatives of w associated with the linearized Hamiltonian H:

$$w_1 = A_\lambda w, \quad w_2 = A^*_\lambda w_1, \quad w_3 = A_\lambda w_2$$

which satisfy from (3.27):

$$\partial_t w_1 + \tilde{H}_\lambda w_1 = \frac{\partial_t Z_\lambda}{r} w + A_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)$$

$$\partial_t w_2 + H_\lambda w_2 = \frac{\partial_t V_\lambda}{r^2} w + H_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right) \tag{3.34}$$

$$\partial_t w_3 + \tilde{H}_\lambda w_3 = \frac{\partial_t Z_\lambda}{r} w_2 + A_\lambda\left(\frac{\partial_t V_\lambda}{r^2} w\right) + (AH)_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right) \tag{3.35}$$

We similarly use in the following steps the notation:

$$\varepsilon_1 = A\varepsilon, \quad \varepsilon_2 = A^*\varepsilon_1, \quad \varepsilon_3 = A\varepsilon_2.$$

Observe from (3.9) that

$$\mathcal{E}_4 = \int |A^*\varepsilon_3|^2. \tag{3.36}$$

We recall the action of time derivatives on rescaling:

$$\partial_t v_\lambda = \frac{1}{\lambda^2}\left(\partial_s v - \frac{\lambda_s}{\lambda}\Lambda v\right)_\lambda.$$

step 2 Modified energy identity.

We compute the energy identity on (3.35) using (3.26):

$$\begin{aligned}
\frac{1}{2}\frac{d\mathcal{E}_4}{dt} &= \frac{1}{2}\frac{d}{dt}\left\{\int \tilde{H}_\lambda w_3 w_3\right\} = \int \tilde{H}_\lambda w_3 \partial_t w_3 + \int \frac{\partial_t \tilde{V}_\lambda}{2r^2} w_3^2 \\
&= -\int (\tilde{H}_\lambda w_3)^2 + b\int \frac{(\Lambda\tilde{V})_\lambda}{2\lambda^2 r^2} w_3^2 - \left(\frac{\lambda_s}{\lambda} + b\right)\int \frac{(\Lambda\tilde{V})_\lambda}{2\lambda^2 r^2} w_3^2 \\
&+ \int \tilde{H}_\lambda w_3 \left[\frac{\partial_t Z_\lambda}{r} w_2 + A_\lambda\left(\frac{\partial_t V_\lambda}{r^2} w\right) + (AH)_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\right].
\end{aligned} \tag{3.37}$$

We now aim at using the dissipative term $\int(\tilde{H} w_3)^2$ to treat the quadratic terms in the RHS of (3.37). Observe however that this quantity is delicate to use because it is positive but not coercive a priori[20]. Also one can explicitly compute for the sphere target

$$b\frac{(\Lambda V)_\lambda}{\lambda^2 r^2} = -\frac{b}{\lambda^8}\frac{4}{(1+y^2)^2} < 0,$$

and thus the critical in size quadratic term has the right sign in this case:

$$b\int \frac{(\Lambda\tilde{V})_\lambda}{\lambda^2 r^2} w_3^2 < 0$$

which would allow some simplification of our analysis. However, this sign property does not seem to hold a priori for the general metric g we consider. We nevertheless claim using a similar algebra as in [22], [19] that this term can be treated thanks to a

[20] as can be seen by considering the zero of $\tilde{H}$ given $w_3 = \frac{1}{y\Lambda Q}\int_0^y \tau(\Lambda Q)^2 d\tau$.

further integration by parts in time which in the dispersive cases would correspond to a Morawetz type computation. Indeed, we compute from (3.34), (3.35):

$$\begin{aligned}\frac{d}{dt}\left\{\int\frac{b(\Lambda Z)_\lambda}{\lambda^2 r}w_3w_2\right\} &= \int\frac{d}{dt}\left(\frac{b(\Lambda Z)_\lambda}{\lambda^2 r}\right)w_3w_2\\ &+ \int\frac{b(\Lambda Z)_\lambda}{\lambda^2 r}w_2\left[-\tilde{H}_\lambda w_3+\frac{\partial_t Z_\lambda}{r}w_2+A_\lambda\left(\frac{\partial_t V_\lambda}{r^2}w\right)+(AH)_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\right]\\ &+ \int\frac{b(\Lambda Z)_\lambda}{\lambda^2 r}w_3\left[-A^*_\lambda w_3+\frac{\partial_t V_\lambda}{r^2}w+H_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\right]\end{aligned}$$

We now integrate by parts to compute using (2.8):

$$\begin{aligned}\int\frac{b(\Lambda Z)_\lambda}{\lambda^2 r}w_3A^*_\lambda w_3 &= \frac{b}{\lambda^8}\int\frac{\Lambda Z}{y}\varepsilon_3A^*\varepsilon_3=\frac{b}{\lambda^8}\int\frac{2(1+Z)\Lambda Z-\Lambda^2 Z}{2y^2}\varepsilon_3^2\\ &= \frac{b}{\lambda^8}\int\frac{\Lambda\tilde{V}}{2y^2}\varepsilon_3^2=b\int\frac{(\Lambda\tilde{V})_\lambda}{2\lambda^2r^2}w_3^2.\end{aligned}$$

Injecting this into the energy identity (3.37) yields the modified energy identity:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{\mathcal{E}_4+2\int\frac{b(\Lambda Z)_\lambda}{\lambda^2 r}w_3w_2\right\}=-\int(\tilde{H}_\lambda w_3)^2\\ &- \left(\frac{\lambda_s}{\lambda}+b\right)\int\frac{(\Lambda\tilde{V})_\lambda}{2\lambda^2r^2}w_3^2+\int\frac{d}{dt}\left(\frac{b(\Lambda Z)_\lambda}{\lambda^2 r}\right)w_3w_2\\ &+ \int\tilde{H}_\lambda w_3\left[\frac{\partial_t Z_\lambda}{r}w_2+\int A_\lambda\left(\frac{\partial_t V_\lambda}{r^2}w\right)+(AH)_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\right]\\ &+ \int\frac{b(\Lambda Z)_\lambda}{\lambda^2 r}w_2\left[-\tilde{H}_\lambda w_3+\frac{\partial_t Z_\lambda}{r}w_2+A_\lambda\left(\frac{\partial_t V_\lambda}{r^2}w\right)+(AH)_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\right]\\ &+ \int\frac{b(\Lambda Z)_\lambda}{\lambda^2 r}w_3\left[\frac{\partial_t V_\lambda}{r^2}w+H_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\right]\end{aligned}\tag{3.38}$$

We now aim at estimating all terms in the RHS of (3.38). All along the proof, we shall make an implicit use of the coercivity estimates of Lemma B.1 and Lemma B.3.

step 3 Lower order quadratic terms.

We start with treating the lower order quadratic terms in (3.38) *using dissipation.* Indeed, we have from (2.14), (2.15), (3.30) the bounds:

$$|\partial_t Z_\lambda|+|\partial_t V_\lambda|\lesssim\frac{b}{\lambda^2}\left(|\Lambda Z|+|\Lambda V|\right)_\lambda\lesssim\frac{b}{\lambda^2}\frac{y^2}{1+y^4}\tag{3.39}$$

and thus from Cauchy Schwartz, the rough bound (3.30) and Lemma B.3:

$$\begin{aligned}&\int\left|\tilde{H}_\lambda w_3\left[\frac{\partial_t Z_\lambda}{r}w_2+\int A_\lambda\left(\frac{\partial_t V_\lambda}{r^2}w\right)\right]\right|+\int|\tilde{H}_\lambda w_3|\left|\frac{b(\Lambda Z)_\lambda}{\lambda^2 r}w_2\right|\\ \leq\;&\frac{1}{2}\int|\tilde{H}_\lambda w_3|^2+\frac{b^2}{\lambda^8}\left[\int\frac{\varepsilon_2^2}{1+y^6}+\int\frac{\varepsilon_1^2}{1+y^8}+\int\frac{\varepsilon^2}{y^2(1+y^8)}\right]\\ \leq\;&\frac{1}{2}\int|\tilde{H}_\lambda w_3|^2+\frac{b}{\lambda^8}b|\log b|^Cb^4.\end{aligned}$$

All other quadratic terms are lower order by a factor b using again (3.30) and Lemma B.3:

$$\begin{aligned}
& \left|\frac{\lambda_s}{\lambda}+b\right|\int\left|\frac{(\Lambda\tilde{V})_\lambda}{2\lambda^2r^2}w_3^2\right|+\int\left|\frac{b(\Lambda Z)_\lambda}{\lambda^2 r}w_2\left[\frac{\partial_t Z_\lambda}{r}w_2+A_\lambda\left(\frac{\partial_t V_\lambda}{r^2}w\right)\right]\right| \\
+ & \int\left|\frac{b(\Lambda Z)_\lambda}{\lambda^2 r}w_3\frac{\partial_t V_\lambda}{r^2}w\right|+\left|\int\frac{d}{dt}\left(\frac{b(\Lambda Z)_\lambda}{\lambda^2 r}\right)w_3w_2\right| \\
\lesssim & \frac{b^2}{\lambda^8}\left[\int\frac{\varepsilon_3^2}{1+y^2}+\int\frac{\varepsilon_2^2}{1+y^4}+\int\frac{\varepsilon_1^2}{1+y^8}+\int\frac{\varepsilon^2}{y^2(1+y^8)}\right] \\
\lesssim & \frac{b}{\lambda^8}b|\mathrm{log} b|^C b^4.
\end{aligned}$$

We similarly estimate the boundary term in time:

$$\left|\int\frac{b(\Lambda Z)_\lambda}{\lambda^2 r}w_3w_2\right|\lesssim\frac{b}{\lambda^6}\left[\int\frac{\varepsilon_3^2}{1+y^2}+\int\frac{\varepsilon_2^2}{1+y^4}\right]\lesssim\frac{b}{\lambda^6}|\mathrm{log} b|^C b^4.$$

We inject these estimates into (3.38) to derive the preliminary bound:

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\left\{\frac{1}{\lambda^6}\left[\mathcal{E}_4+O\left(\sqrt{b}\frac{b^4}{|\mathrm{log} b|^2}\right)\right]\right\}\leq-\frac{1}{2}\int(\tilde{H}_\lambda w_3)^2+\frac{b}{\lambda^8}\frac{b^4}{|\mathrm{log} b|^2} \qquad (3.40)\\
+ & \int\tilde{H}_\lambda w_3 A_\lambda H_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)+\int H_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\left[\frac{b(\Lambda Z)_\lambda}{\lambda^2 r}w_3+A^*_\lambda\left(\frac{b(\Lambda Z)_\lambda}{\lambda^2 r}w_2\right)\right]
\end{aligned}$$

with constants independent of M for $|b|<b^*(M)$ small enough. We now aim at estimating all terms in the RHS of (3.40).

step 4 Further use of dissipation.

Let us introduce the decomposition from (3.22), (3.23):

$$\mathcal{F}=\mathcal{F}_0+\mathcal{F}_1,\quad \mathcal{F}_0=-\tilde{\Psi}_b-Mod(t),\quad \mathcal{F}_1=L(\varepsilon)-N(\varepsilon).$$

The first term in the RHS of (3.40) is estimated after an integration by parts:

$$\begin{aligned}
& \left|\int\tilde{H}_\lambda w_3A_\lambda H_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\right|\leq\frac{C}{\lambda^8}\|A^*\varepsilon_3\|_{L^2}\|H^2\mathcal{F}_0\|_{L^2}+\frac{1}{4}\int(\tilde{H}_\lambda w_3)^2+\frac{C}{\lambda^8}\int|AH\mathcal{F}_1|^2\\
\leq & \frac{C}{\lambda^8}\left[\|H^2\mathcal{F}_0\|_{L^2}\sqrt{\mathcal{E}_4}+\|AH\mathcal{F}_1\|^2_{L^2}\right]+\frac{1}{4}\int(\tilde{H}_\lambda w_3)^2 \qquad (3.41)
\end{aligned}$$

for some universal constant $C>0$ independent of M. The last two terms in (3.40) can be estimated in brute force from Cauchy Schwarz:

$$\begin{aligned}
\left|\int H_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\frac{b(\Lambda Z)_\lambda}{\lambda^2 r}w_3\right| & \lesssim \frac{b}{\lambda^8}\left(\int\frac{1+|\mathrm{log} y|^2}{1+y^4}|H\mathcal{F}|^2\right)^{\frac12}\left(\int\frac{\varepsilon_3^2}{y^2(1+|\mathrm{log} y|^2)}\right)^{\frac12}\\
& \lesssim \frac{b}{\lambda^8}\sqrt{\mathcal{E}_4}\left(\int\frac{1+|\mathrm{log} y|^2}{1+y^4}|H\mathcal{F}|^2\right)^{\frac12} \qquad (3.42)
\end{aligned}$$

where constants are independent of M thanks to the estimate (B.1) for ε_3. Similarly:

$$\begin{aligned}
&\left|\int H_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)A_\lambda^*\left(\frac{b(\Lambda Z)_\lambda}{\lambda^2 r}w_2\right)\right|\\
\lesssim\;& \frac{b}{\lambda^8}\left(\int\frac{1+|\mathrm{log}y|^2}{1+y^2}|AH\mathcal{F}|^2\right)^{\frac12}\left(\int\frac{\varepsilon_2^2}{1+y^4(1+|\mathrm{log}y|^2)}\right)^{\frac12}\\
\lesssim\;& \frac{b}{\lambda^8}C(M)\sqrt{\mathcal{E}_4}\left(\int\frac{1+|\mathrm{log}y|^2}{1+y^2}|AH\mathcal{F}_0|^2+\int|AH\mathcal{F}_1|^2\right)^{\frac12}. \qquad (3.43)
\end{aligned}$$

We now claim the bounds:

$$\int\frac{1+|\mathrm{log}y|^2}{1+y^4}|H\mathcal{F}|^2\lesssim\frac{b^4}{|\mathrm{log}b|^2}+\frac{\mathcal{E}_4}{\mathrm{log}M}, \qquad (3.44)$$

$$\int\frac{1+|\mathrm{log}y|^2}{1+y^2}|AH\mathcal{F}_0|^2\lesssim\delta^*\left(\frac{b^4}{|\mathrm{log}b|^2}+\mathcal{E}_4\right), \qquad (3.45)$$

$$\int|H^2\mathcal{F}_0|^2\lesssim b^2\left[\frac{b^4}{|\mathrm{log}b|^2}+\frac{\mathcal{E}_4}{\mathrm{log}M}\right], \qquad (3.46)$$

$$\int|AH\mathcal{F}_1|^2\lesssim b\left[\frac{b^4}{|\mathrm{log}b|^2}+\frac{\mathcal{E}_4}{\mathrm{log}M}\right], \qquad (3.47)$$

with all $\lesssim$ constants independent of M for $|b|<b^*(M)$ small enough, and where

$$\delta^*=\delta^*(b^*(M))\to 0 \quad\text{as}\quad b^*(M)\to 0.$$

Injecting these bounds together with (3.41), (3.42), (3.43) into (3.40) concludes the proof of (3.33). We now turn to the proof of (3.44), (3.45), (3.46), (3.47).

step 5 $\tilde{\Psi}_b$ terms.

The contribution of $\tilde{\Psi}_b$ terms to (3.44), (3.45), (3.46) is estimated from (2.62), (2.63), (2.64) which are at the heart of the construction of $\tilde{Q}_b$ and yield the desired bounds.

step 6 $Mod(t)$ terms.

Recall the definition (2.60) of $Mod(t)$:

$$Mod(t)=-\left(\frac{\lambda_s}{\lambda}+b\right)\Lambda\tilde{Q}_b+(b_s+b^2)(\tilde{T}_1+2b\tilde{T}_2).$$

For (3.44), we estimate using the rough bounds (2.46), (2.51) and the control of the modulation parameters (3.28), (3.29) to estimate:

$$\begin{aligned}
&\int\frac{1+|\mathrm{log}y|^2}{1+y^4}|HMod|^2\\
\lesssim\;& \left|\frac{\lambda_s}{\lambda}+b\right|^2\int\frac{1+|\mathrm{log}y|^2}{1+y^4}|H\Lambda\tilde{Q}_b|^2+|b_s+b^2|^2\int\frac{1+|\mathrm{log}y|^2}{1+y^4}|H(\tilde{T}_1+2b\tilde{T}_2)|^2\\
\lesssim\;& \left|\frac{\lambda_s}{\lambda}+b\right|^2+|b_s+b^2|^2\lesssim\frac{b^4}{|\mathrm{log}b|^2}+\frac{\mathcal{E}_4}{\mathrm{log}M}.
\end{aligned}$$

For (3.45), we use the cancellations $H\Lambda Q=0$, $AHT_1=0$ and the rough bounds (2.46), (2.51) to derive the degenerate bounds:

$$\begin{aligned}\int \frac{1+|\text{log}y|^2}{1+y^2}|AH\Lambda\tilde{Q}_b|^2 &\lesssim \int_{y\leq 2B_1}\frac{1+|\text{log}y|^2}{1+y^2}\left|\frac{by|\text{log}y|+b^2(1+y^3)+b^3(1+y^5)}{1+y^3}\right|^2\\ &\lesssim b^2\end{aligned}$$

$$\begin{aligned}\int \frac{1+|\text{log}y|^2}{1+y^2}|AH(\tilde{T}_1+b\tilde{T}_2)|^2 &\lesssim \int \frac{1+|\text{log}y|^2}{1+y^2}\left|\frac{y\text{log}y}{y^3}\mathbf{1}_{B_1\leq y\leq 2B_1}+b\frac{1+y^3}{1+y^3}\mathbf{1}_{y\leq 2B_1}\right|^2\\ &\lesssim b\end{aligned}$$

and thus from (3.28), (3.29):

$$\int \frac{1+|\text{log}y|^2}{1+y^2}|AH\mathcal{M}od|^2\lesssim b\left[\left(\frac{\lambda_s}{\lambda}+b\right)^2+(b_s+b^2)^2\right]\lesssim b\left[\frac{b^4}{|\text{log}b|^2}+\frac{\mathcal{E}_4}{\text{log}M}\right].$$

For (3.46), we estimate from the rough bounds (2.46), (2.51):

$$\int |H^2\Lambda\tilde{Q}_b|^2\lesssim\int\left|\frac{by\text{log}y+b^2(1+y^3)+b^3(1+y^5)}{1+y^4}\right|^2\lesssim b^2,$$

$$\int |H^2\tilde{T}_1|^2\lesssim\int_{B_1\leq y\leq 2B_1}\left|\frac{y\text{log}y}{y^4}\right|^2\lesssim\frac{|\text{log}b|^2}{B_1^4}\lesssim b^2.$$

The last term is more subtle and we claim:

$$\int |H^2\tilde{T}_2|^2\lesssim 1 \qquad (3.48)$$

which yields

$$\int |H^2Mod|^2\lesssim b^2\left[\left(\frac{\lambda_s}{\lambda}+b\right)^2+(b_s+b^2)^2\right]\lesssim b^2\left[\frac{b^4}{|\text{log}b|^2}+\frac{\mathcal{E}_4}{|\text{log}M}\right]$$

and concludes the proof of (3.46).

Proof of (3.48): First by definition of $\tilde{T}_2=\chi_{B_1}T_2$, the rough bound (2.46) and (2.44):

$$\int |H^2\tilde{T}_2|^2\lesssim\left[\int_{B_1\leq y\leq 2B_1}\left|\frac{y^3}{y^4}\right|^2+\int_{y\leq 2B_1}|H\Sigma_2|^2\right]\lesssim 1+\int_{y\leq 2B_1}|H\Sigma_2|^2. \qquad (3.49)$$

We now compute from (2.41), (2.34):

$$\begin{aligned}H\Sigma_2 &= H\Sigma_b+H(T_1-\Lambda T_1)+O\left(\frac{y^2|\text{log}y|^2}{1+y^5}\right)\\ &= -c_b\chi_{\frac{B_0}{4}}\Lambda Q+d_bH[(1-\chi_{3B_0})\Lambda Q]+H(T_1-\Lambda T_1)+O\left(\frac{y^2|\text{log}y|^2}{1+y^5}\right)\\ &= \frac{1}{|\text{log}b|}O\left(\frac{1}{1+y}\mathbf{1}_{y\leq 3B_0}\right)+H(T_1-\Lambda T_1)+O\left(\frac{y^2|\text{log}y|^2}{1+y^5}\right)\end{aligned}$$

and observe using (2.54), the asymptotics (2.3) of ΛQ and the fundamental cancellation $(\Lambda+\Lambda^2)\left(\frac{1}{y}\right)=0$ that:

$$\begin{aligned}HT_1-H\Lambda T_1 &= HT_1-\left(2HT_1+\Lambda HT_1-\frac{\Lambda V}{y^2}T_1\right)=\Lambda Q+\Lambda^2Q+O\left(\frac{y\text{log}y}{1+y^4}\right)\\ &= O\left(\frac{\text{log}y}{1+y^3}\right).\end{aligned}$$

We thus conclude:

$$\int |H\Sigma_2|^2 \lesssim \frac{1}{|\log b|^2}\int_{y\leq 2B_0}\frac{1}{1+y^2}+\int_{y\leq 2B_1}\frac{|\log y|^4}{1+y^6}\lesssim 1$$

which together with (3.49) concludes the proof of (3.48).

step 7 Small linear term $L(\varepsilon)$.

Let us rewrite from a Taylor expansion:

$$L(\varepsilon)=-\frac{N_2(\tilde{\alpha})}{y^2}\varepsilon \ \text{ with } \ N_2(\tilde{\alpha})=f'(Q+\tilde{\alpha})-f'(Q)=\tilde{\alpha}\int_0^1 f''(Q+\tau\tilde{\alpha})d\tau. \quad (3.50)$$

Near the origin $y\leq 1$, we use $f''(0)=0$ and the estimate $|\tilde{\alpha}|\lesssim by^3$ near the origin by construction to obtain the high order cancellation

$$N_2(\tilde{\alpha})\lesssim |\tilde{\alpha}|y\lesssim by^4 \quad (3.51)$$

which together with the bounds (B.10), (B.11), (B.12) easily yields:

$$\int_{y\leq 1}\frac{1+|\log y|^2}{1+y^4}|HL(\varepsilon)|^2+\int_{y\leq 1}|AHL(\varepsilon)|^2\lesssim b^6.$$

For $y\geq 1$, we use $f''(\pi)=0$ and the bounds (2.45), (2.50) to derive the bound:

$$\begin{aligned}|N_2(\tilde{\alpha})| &\lesssim |\tilde{\alpha}|\left[\frac{1}{1+y}+|\tilde{\alpha}|\right]\lesssim \left[b|\log y|+b^2y^2|\log y|^2\right]\mathbf{1}_{y\leq 2B_1}\\ &\lesssim b|\log b|^C\mathbf{1}_{y\leq 2B_1}. \end{aligned} \quad (3.52)$$

A brute force computation taking further derivatives and using (B.5) now yields the control:

$$\begin{aligned}\int_{y\geq 1}|AHL(\varepsilon)|^2+\int_{y\geq 1}\frac{1+|\log y|^2}{1+y^4}|HL(\varepsilon)|^2 &\lesssim b^2|\log b|^C\int_{y\leq 2B_1}\Sigma_{i=0}^3\frac{|\partial_y^i\varepsilon|^2}{y^{2(4-i)}}\\ &\lesssim b^2|\log b|^C\mathcal{E}_4\lesssim b\delta^*\mathcal{E}_4.\end{aligned}$$

This concludes the proof of (3.44), (3.47) for $L(\varepsilon)$ terms.

step 8 Nonlinear term $N(\varepsilon)$.

Let us now treat the nonlinear term (3.25). We split the contribution at the origin and far out and claim:

$$\forall y\leq 1, \ \ |AHN(\varepsilon)(y)|\lesssim C(M)b^4|\log y|^4, \quad (3.53)$$

$$\int_{y\geq 1}(AHN(\varepsilon))^2\lesssim C(M)\frac{b^6}{|\log b|^2}. \quad (3.54)$$

which implies (3.47) for $N(\varepsilon)$. The estimate (3.44) follows along similar lines and is in fact simpler and left to the reader.

Proof of (3.53): We need to treat the possible singularity at the origin. For this, let us rewrite by Taylor expansion:

$$N(\varepsilon)=z^2N_0(\varepsilon) \ \text{ with } \ z=\frac{\varepsilon}{y}, \ \ N_0(\varepsilon)=\int_0^1(1-\tau)f''(\tilde{Q}_b+\tau\varepsilon)d\tau,$$

and thus:

$$H(N(\varepsilon))=-N_0(\varepsilon)\Delta(z^2)-2z\partial_y z\partial_y N_0(\varepsilon)+z^2H(N_0(\varepsilon)),$$

$$\begin{aligned} AHN(\varepsilon) &= -A(N_0(\varepsilon))\Delta(z^2) + N_0(\varepsilon)\partial_y\Delta(z^2) - 2A(z\partial_y z)\partial_y(N_0(\varepsilon)) \\ &+ 2z\partial_y z\partial_{yy}(N_0(\varepsilon)) + z^2 AH(N_0(\varepsilon)) - 2z\partial_y zH(N_0(\varepsilon)). \end{aligned} \quad (3.55)$$

We now use the Hardy bounds (B.10), (B.11), (B.12) and the degeneracy $|Z-1|+|V-1|\lesssim y^2$ for $y\leq 1$ to estimate for $0<y\leq 1$:

$$|\partial_{yy}\varepsilon| = \left|-H\varepsilon + \frac{A\varepsilon}{y} + \frac{1-Z+V-1}{y^2}\varepsilon\right| \lesssim b^2 y|\text{log}y|,$$

$$|z| = \left|\frac{\varepsilon}{y}\right| \lesssim b^2,$$

$$|\partial_y z| = \left|-\frac{A\varepsilon}{y} + \frac{Z-1}{y^2}\varepsilon\right| \lesssim b^2 y|\text{log}y|,$$

$$|\partial_{yy} z| = \left|\frac{2+Z}{y^2}A\varepsilon - \frac{H\varepsilon}{y} + \partial_y\left(\frac{Z-1}{y^2}\varepsilon\right)\right| \lesssim b^2|\text{log}y|,$$

$$\begin{aligned} |\partial_{yyy} z| &= \left|\partial_y\left(\frac{2+Z}{y^2}\right)A\varepsilon + \frac{2+Z}{y^2}\left(H\varepsilon - \frac{1+Z}{y}A\varepsilon\right) + \frac{H\varepsilon}{y^2}\right. \\ &- \left.\frac{1}{y}\left(-AH\varepsilon + \frac{Z}{y}H\varepsilon\right) + \partial_{yy}\left(\frac{Z-1}{y^2}\varepsilon\right)\right| \\ &\lesssim \frac{b^2|\text{log}y|}{y} \end{aligned}$$

with constants depending on M. We now estimate using $f^{(2k)}(0)=0$: for $0<y\leq 1$,

$$|N_0(\varepsilon)| \lesssim y,$$

$$|\partial_y N_0(\varepsilon)| = \left|\int_0^1 (1-\tau)\partial_y(\tilde{Q}_b+\tau\varepsilon)f^{(3)}(\tilde{Q}_b+\tau\varepsilon)d\tau\right| \lesssim 1,$$

$$\begin{aligned} |\partial_{yy}N_0(\varepsilon)| &= \left|\int_0^1 (1-\tau)\left[\partial_{yy}(\tilde{Q}_b+\tau\varepsilon)f^{(3)}(\tilde{Q}_b+\tau\varepsilon) + \left(\partial_y(\tilde{Q}_b+\tau\varepsilon)\right)^2 f^{(4)}(\tilde{Q}_b+\tau\varepsilon)\right]d\tau\right| \\ &\lesssim y|\text{log}y|^2 \end{aligned}$$

We need to exploit further cancellations for $AN_0(\varepsilon)$ and this requires pushing the Taylor expansion:

$$N_0(\varepsilon) = \frac{1}{2}f''(\tilde{Q}_b) + \varepsilon N_1(\varepsilon), \quad N_1(\varepsilon) = \int_0^1\int_0^1 (1-\sigma)\tau(1-\tau)f^{(3)}(\tilde{Q}_b+\sigma\tau\varepsilon)d\sigma d\tau.$$

By construction, $\tilde{Q}_b$ is a smooth function at the origin and admits a Taylor expansion

$$\tilde{Q}_b = c_1(b)y + c_2(b)y^3 + O(y^5) \quad \text{with} \quad |c_1(b), c_2(b)| \lesssim 1$$

and hence:

$$|Af''(\tilde{Q}_b)| + |Hf''(\tilde{Q}_b)| + |AH(f''(\tilde{Q}_b))| \lesssim 1.$$

We therefore estimate arguing like for $N_0(\varepsilon)$:

$$|A(\varepsilon N_1(\varepsilon))| = |A\varepsilon N_1(\varepsilon) - \varepsilon\partial_y(N_1(\varepsilon))| \lesssim y^2|\text{log}y|,$$

$$|H(\varepsilon N_1(\varepsilon))| = |N_1(\varepsilon)H\varepsilon - 2\partial_y\varepsilon\partial_y N_1(\varepsilon) - \varepsilon\partial_{yy}N_1(\varepsilon)| \lesssim y|\text{log}y|^2,$$

$$\begin{aligned} |AH(\varepsilon N_1(\varepsilon))| &= |N_1(\varepsilon)AH\varepsilon - (H\varepsilon)\partial_y N_1(\varepsilon) - 2(A\partial_y\varepsilon)\partial_y N_1(\varepsilon) + 2\partial_y\varepsilon\partial_{yy}N_1(\varepsilon) \\ &- A\varepsilon\partial_{yy}N_1(\varepsilon) + \varepsilon\partial_{yyy}N_1(\varepsilon)| \\ &\lesssim |\text{log}y|^4 \end{aligned}$$

Injecting the collection of above estimates into (3.55) now yields (3.53).
Proof of (3.54): For $y \geq 1$, we estimate from (B.13), (B.14):

$$\|z\|_{L^\infty(y\geq 1)} \lesssim b|\log b|^C, \quad \|\partial_y z\|_{L^\infty(y\geq 1)} + \|\frac{z}{y}\|_{L^\infty(y\geq 1)} \lesssim b^{\frac{3}{2}}|\log b|^C.$$

The construction of $\tilde{Q}_b$ yields the bounds for $y \geq 1$:

$$|\partial_y \tilde{Q}_b| \lesssim |\log b|^C \left(\frac{1}{y^2} + b\mathbf{1}_{y\leq 2B_1}\right),$$

$$|\partial_{yy}\tilde{Q}_b| \lesssim |\log b|^C \left(\frac{1}{y^3} + \frac{b}{y}\mathbf{1}_{y\leq 2B_1}\right) \lesssim |\log b|^C \left(\frac{1}{y^3} + b^{\frac{3}{2}}\right),$$

$$|\partial_{yyy}\tilde{Q}_b| \lesssim |\log b|^C \left(\frac{1}{y^4} + \frac{b}{y^2}\mathbf{1}_{y\leq 2B_1}\right) \lesssim |\log b|^C \left(\frac{1}{y^4} + b^2\right)$$

which together with (B.13), (B.14), (B.15) yields the pointwise bounds:

$$|N_0(\varepsilon)| \lesssim 1, \tag{3.56}$$

$$|\partial_y N_0(\varepsilon)| \lesssim |\log b|^C \left(\frac{1}{y^2} + b + \|\partial_y \varepsilon\|_{L^\infty(y\geq 1)}\right) \lesssim |\log b|^C \left(\frac{1}{y^2} + b\right), \tag{3.57}$$

$$|\partial_{yy} N_0(\varepsilon)| \lesssim |\log b|^C \left[\frac{1}{y^3} + b^{\frac{3}{2}} + \|\partial_{yy}\varepsilon\|_{L^\infty(y\geq 1)}\right] \lesssim |\log b|^C \left[\frac{1}{y^3} + b^{\frac{3}{2}}\right], \tag{3.58}$$

$$|\partial_{yyy} N_0(\varepsilon)| \lesssim |\log b|^C \left[\frac{1}{y^4} + b^2 + \|\partial_{yyy}\varepsilon\|_{L^\infty(y\geq 1)}\right] \lesssim |\log b|^C \left[\frac{1}{y^4} + b^2\right]. \tag{3.59}$$

We now compute:

$$H(N(\varepsilon)) = N_0(\varepsilon)H(z^2) - 2z\partial_y z\partial_y N_0(\varepsilon) - z^2\Delta(N_0(\varepsilon)), \tag{3.60}$$

$$\begin{aligned} AHN(\varepsilon) &= N_0(\varepsilon)AH(z^2) - \partial_y N_0(\varepsilon)H(z^2) - 2A(z\partial_y z)\partial_y(N_0(\varepsilon)) \\ &+ 2z\partial_y z\partial_{yy}(N_0(\varepsilon)) - A(z^2)\Delta(N_0(\varepsilon)) + z^2\partial_y\Delta(N_0(\varepsilon)), \end{aligned}$$

and hence using the L^2 weighted bounds (B.5), (B.6), (B.7) and the L^∞ bounds (B.13), (B.14):

$$\begin{aligned} \int_{y\geq 1} |N_0(\varepsilon)AH(z^2)|^2 &\lesssim \int_{y\geq 1}\Big[|\partial_y z\Delta z|^2 + |z\partial_y\Delta z|^2 + |\partial_y z\partial_{yy}z|^2 + \frac{|z\partial_y z|^2}{y^4} \\ &+ \frac{1}{y^2}\left(|z\Delta z|^2 + |\partial_y z|^4 + \frac{|z|^4}{y^4}\right)\Big] \\ &\lesssim b^6|\log b|^C, \end{aligned}$$

$$\int_{y\geq 1} |\partial_y N_0(\varepsilon)H(z^2)|^2 \lesssim \int_{y\geq 1}\left(\frac{1}{y^2} + b^2|\log b|^C\right)\left(|z|^2|\Delta z|^2 + |\partial_y z|^4 + \frac{|z|^4}{y^4}\right) \lesssim b^6|\log b|^C,$$

$$\begin{aligned} \int_{y\geq 1} |A(z\partial_y z)\partial_y N_0(\varepsilon)|^2 &\lesssim \int_{y\geq 1}\left(\frac{1}{y^2} + b^2|\log b|^C\right)\left(|z|^2|\partial_{yy}z|^2 + |\partial_y z|^4 + \frac{|z|^2|\partial_y z|^2}{y^2}\right) \\ &\lesssim b^6|\log b|^C, \end{aligned}$$

$$\int_{y\geq 1} |z\partial_y z\partial_{yy}N_0(\varepsilon)|^2 \lesssim |\log b|^C \int_{y\geq 1} |z|^2|\partial_y z|^2\left[\frac{1}{y^6} + b^3\right] \lesssim b^6|\log b|^C,$$

$$\int_{y\geq 1} |A(z^2)\Delta N_0(\varepsilon)|^2 \lesssim |\log b|^C \int_{y\geq 1}\left[|z|^2|\partial_y z|^2 + \frac{|z|^4}{y^2}\right]\left[\frac{1}{y^6} + b^3\right] \lesssim b^6|\log b|^C,$$

$$\int_{y\geq 1} |z^2\partial_y\Delta(N_0(\varepsilon))|^2 \lesssim |\log b|^C \int_{y\geq 1} |z|^4\left[\frac{1}{y^8} + b^4\right] \lesssim b^6|\log b|^C.$$

This concludes the proof of (3.54).
This concludes the proof of (3.44), (3.45), (3.46), (3.47) and thus of Proposition 3.4.

4. Sharp description of the singularity formation

In this section, we start with completing the proof of the bootstrap Proposition 3.1. Theorem 1.1 will then easily follow.

4.1. **Closing the bootstrap.** We are now in position to close the boostrap bounds of Proposition 3.1.

Proof of Proposition 3.1

step 1 Energy bound.

First observe that (3.16) and the modulation equation (3.29) ensure for M large enough[21]:

$$b_s \leq 0$$

and the upper bound in (3.17) follows. We now claim:

$$b(t) > 0 \quad \text{on} \quad [0, T_1).$$

Indeed, from (3.30), if $b(t_0) = 0$ for some $t_0 \in [0, T_1)$, then $b(t) \equiv 0$ on some $[t_0 - \delta, t_0]$ and thus from (3.30), (3.16), $\lambda(t) \equiv \lambda(t_0)$ and $u(t) \equiv Q_{\lambda(t_0)}$ on $[t_0 - \delta, t_0]$. We can thus iterate on $\delta > 0$ and conclude that $u(0)$ is initially a harmonic map, a contradiction. This concludes the proof of (3.17).
We now prove (3.19) which follows from the conservation of energy. Indeed, let

$$\tilde{\varepsilon} = \varepsilon + \tilde{\alpha},$$

then

$$\begin{aligned} E_0 &= \int |\partial_y (Q + \tilde{\varepsilon})|^2 + \int \frac{g^2(Q + \tilde{\varepsilon})}{y^2} \\ &= E(Q) + (H\varepsilon, \varepsilon) + \int \frac{1}{y^2} \left[g^2(Q + \tilde{\varepsilon}) - 2f(Q)\varepsilon - f'(Q)\tilde{\varepsilon}^2 \right]. \end{aligned} \tag{4.1}$$

We now recall from Lemma B.1 and Lemma B.2 the coercivity properties:

$$(H\varepsilon, \varepsilon) \geq c(M) \left[\int |\partial_y \varepsilon|^2 + \int \frac{|\varepsilon|^2}{y^2} \right],$$

$$(H\tilde{\varepsilon}, \tilde{\varepsilon}) \geq c(M) \left[\int |\partial_y \tilde{\varepsilon}|^2 + \int \frac{|\tilde{\varepsilon}|^2}{y^2} \right] + O(C(M) b^2).$$

The nonlinear term is estimated from a Taylor expansion:

$$\left| \int \frac{1}{y^2} \left[g^2(Q + \tilde{\varepsilon}) - 2f(Q)\tilde{\varepsilon} - f'(Q)\tilde{\varepsilon}^2 \right] \right| \lesssim \int \frac{|\tilde{\varepsilon}|^3}{y^2} \lesssim \left(\int |\partial_y \tilde{\varepsilon}|^2 + \int \frac{|\tilde{\varepsilon}|^2}{y^2} \right)^{\frac{3}{2}}.$$

where we used the Sobolev bound

$$\|\tilde{\varepsilon}\|_{L^\infty}^2 \lesssim \|\partial_y \tilde{\varepsilon}\|_{L^2} \|\frac{\tilde{\varepsilon}}{y}\|_{L^2}.$$

[21] recall that K in the bootstrap bounds is large but independent of M.

We inject these bounds into the conservation of energy (4.1) and use the bound on the profile

$$\int |\partial_y \tilde{\alpha}|^2 + \int \frac{|\tilde{\alpha}|^2}{y^2} \lesssim b|\log b|^C$$

and (3.12), (3.17) to estimate:

$$\begin{aligned}\int |\partial_y \varepsilon|^2 + \int \frac{|\varepsilon|^2}{y^2} & \lesssim \int |\partial_y \tilde{\varepsilon}|^2 + \int \frac{|\tilde{\varepsilon}|^2}{y^2} + b|\log b|^C \lesssim C(M)\,|E_0 - E(Q)| + O(C(M)b) \\ & \lesssim C(M)b(0) \leq \sqrt{b(0)}\end{aligned}$$

for $|b(0)| \leq b^*(M)$ small enough, and (3.19) is proved.

step 2 Control of $\mathcal{E}_4$.

We now close the bootstrap bound (3.20) which follows by reintegrating the Lyapounov monotonicity (3.33) in the regime governed by the modulation equations (3.28), (3.29). Indeed, inject the bootstrap bound (3.16) into the monotonicity formula (3.33) and integrate in time; this yields: $\forall t \in [0, T_1)$,

$$\begin{aligned}\mathcal{E}_4(t) & \leq 2\left(\frac{\lambda(t)}{\lambda(0)}\right)^6 \left[\mathcal{E}_4(0) + C\sqrt{b(0)}\frac{b^4(0)}{|\log b(0)|^2}\right] + \frac{b^4(t)}{|\log b(t)|^2} \\ & + C\left[1 + \frac{K}{\log M} + \sqrt{K}\right]\lambda^6(t)\int_0^t \frac{b}{\lambda^8}\frac{b^4}{|\log b|^2}d\tau\end{aligned} \tag{4.2}$$

for some universal constant $C > 0$ independent of M.
Let us now consider two constants

$$\alpha_1 = 2 - \frac{C_1}{\sqrt{\log M}}, \quad \alpha_2 = 2 + \frac{C_2}{\sqrt{\log M}} \tag{4.3}$$

for some large enough universal constants C_1, C_2. We compute using the modulation equations (3.28), (3.29) and the bootstrap bound (3.16):

$$\begin{aligned}\frac{d}{ds}\left\{\frac{|\log b|^{\alpha_i} b}{\lambda}\right\} & = \frac{|\log b|^{\alpha_i}}{\lambda}\left[\left(1 - \frac{\alpha_i}{|\log b|}\right)b_s - \frac{\lambda_s}{\lambda}b\right] \\ & = \frac{|\log b|^{\alpha_i}}{\lambda}\left[\left(1 - \frac{\alpha_i}{|\log b|}\right)b_s + b^2 + O\left(\frac{b^3}{|\log b|}\right)\right] \\ & = \left(1 - \frac{\alpha_i}{|\log b|}\right)\frac{|\log b|^{\alpha_i}}{\lambda}\left[b_s + b^2\left(1 + \frac{\alpha_i}{|\log b|} + O\left(\frac{1}{|\log b|^2}\right)\right)\right] \\ & \left\{\begin{array}{l} \leq 0 \text{ for } i = 1 \\ \geq 0 \text{ for } i = 2. \end{array}\right.\end{aligned}$$

Integrating this from 0 to t yields:

$$\frac{b(0)}{\lambda(0)}\left(\frac{|\log b(0||}{|\log b(t)|}\right)^{\alpha_2} \leq \frac{b(t)}{\lambda(t)} \leq \frac{b(0)}{\lambda(0)}\left(\frac{|\log b(0||}{|\log b(t)|}\right)^{\alpha_1}. \tag{4.4}$$

This yields in particular using the initial bound (3.12) and the bound (3.17):

$$\left(\frac{\lambda(t)}{\lambda(0)}\right)^6 \mathcal{E}_4(0) \leq (b(t)|\log b(t)|^{\alpha_2})^6 \frac{\mathcal{E}_0}{(b(0)|\log b(0)|^{\alpha_2})^6} \leq \frac{b^4(t)}{|\log b(t)|^2}, \tag{4.5}$$

$$C\left(\frac{\lambda(t)}{\lambda(0)}\right)^6\sqrt{b(0)}\frac{b^4(0)}{|\log b(0)|^2} \lesssim \left(\frac{b(t)|\log b(t)|^{\alpha_2}}{b(0)|\log b(0)|^{\alpha_2}}\right)^6\sqrt{b(0)}\frac{b^4(0)}{|\log b(0)|^2}$$
$$\lesssim C(b(t))^{4+\frac14} \leq \frac{b^4(t)}{|\log b(t)|^2}. \quad (4.6)$$

We now compute explicitly using $b=-\lambda\lambda_t+O\left(\frac{b^2}{|\log b|}\right)$ from (3.28):

$$\int_0^t \frac{b}{\lambda^8}\frac{b^4}{|\log b|^2}d\sigma = \frac16\left[\frac{b^4}{\lambda^6|\log b|^2}\right]_0^t - \frac16\int_0^t\frac{b_tb^3}{\lambda^6|\log b|^2}\left(4+\frac{2}{|\log b|}\right)d\tau + O\left(\int_0^t\frac{b}{\lambda^8}\frac{b^5}{|\log b|^2}d\tau\right)$$

which implies using now $|b_s+b^2|\lesssim \frac{b^2}{|\log b|^2}$ from (3.29) and (3.16):

$$\lambda^6(t)\int_0^t\frac{b}{\lambda^8}\frac{b^4}{|\log b|^2}d\sigma \lesssim \left[1+O\left(\frac{1}{|\log b_0|}\right)\right]\frac{b^4(t)}{|\log b(t)|^2}.$$

Injecting this together with (4.5), (4.6) into (4.2) yields

$$\mathcal{E}_4(t)\leq C\frac{b^4(t)}{|\log b(t)|^2}\left[1+\frac{K}{\log M}+\sqrt{K}\right]$$

for some universal constant $C>0$ independent of K and M, and thus (3.20) follows for K large enough independent of M.

step 3 Control of $\mathcal{E}_2$.

We now close the H^2 bound (3.19). This bound is used mostly in the proof of the interpolation estimates of Lemma B.3, and there the power the log is (3.19) is irrelevant. It becomes on the contrary critical in the proof of the regularity (1.14) and this requires being careful with logarithmic growth. For this reason, the profile $\hat{Q}_b$ localized near B_0 given by (2.8) is better adapted to the $\mathcal{E}_2$ control.
Let then the radiation ζ_b given by (2.73) and the new decomposition of the flow:

$$u=(\tilde{Q}_b+\varepsilon)_\lambda=(\hat{Q}_b+\hat{\varepsilon})_\lambda \quad \text{ie} \quad \hat{\varepsilon}=\varepsilon+\zeta_b \quad (4.7)$$

and the renormalization

$$\hat{w}(t,r)=\hat{\varepsilon}(s,y).$$

The equation for $\hat{w}$ is similarly like (3.27):

$$\partial_t\hat{w}+H_\lambda\hat{w}=\frac{1}{\lambda^2}\widehat{\mathcal{F}}_\lambda, \quad \widehat{\mathcal{F}}=-\hat{\Psi}_b-\widehat{Mod}+\hat{L}(\hat{\varepsilon})-\hat{N}(\hat{\varepsilon}), \quad (4.8)$$

$$\hat{L}(\hat{\varepsilon})=\frac{f'(Q)-f'(\hat{Q}_b)}{y^2}\hat{\varepsilon}, \quad \hat{N}(\hat{\varepsilon})=\frac{f(\hat{Q}_b+\hat{\varepsilon})-f(\hat{Q}_b)-\hat{\varepsilon}f'(\hat{Q}_b)}{y^2}.$$

We then let

$$\hat{\varepsilon}_2=H\hat{\varepsilon}, \quad \hat{w}_2=H_\lambda\hat{w},$$

which satisfies from (4.8):

$$\partial_t\hat{w}_2+H_\lambda\hat{w}_2=\frac{\partial_tV_\lambda}{r^2}\hat{w}+H_\lambda\left(\frac{1}{\lambda^2}\widehat{\mathcal{F}}_\lambda\right).$$

We therefore compute the energy identity:

$$\frac{1}{2}\frac{d}{dt}\left\{\int|\hat{w}_2|^2\right\} = \int\hat{w}_2\left[-H_\lambda\hat{w}_2+\frac{\partial_tV_\lambda}{r^2}\hat{w}+H_\lambda\left(\frac{1}{\lambda^2}\widehat{\mathcal{F}}_\lambda\right)\right] \lesssim -\int|A_\lambda\hat{w}_2|^2+\frac{1}{\lambda^4}\left[b\|\hat{\varepsilon}_2\|_{L^2}\left\|\frac{\Lambda V}{y^2}\hat{\varepsilon}\right\|_{L^2}+|(\hat{\varepsilon}_2,H\widehat{\mathcal{F}})|\right] \tag{4.9}$$

and aim at estimating all terms in the above RHS. The local term is estimated using the decomposition (4.7), the estimate (2.75) and Lemma B.3:

$$\int\left|\frac{\Lambda V}{y^2}\hat{\varepsilon}\right|^2\lesssim b^4|\log b|^C+\int\frac{|\zeta_b|^2}{1+y^8}\lesssim b^4|\log b|^C. \tag{4.10}$$

We now claim the bound:

$$|(\hat{\varepsilon}_2,H\widehat{\mathcal{F}})|\lesssim b^3|\log b|^2. \tag{4.11}$$

Asssume (4.11), we then obtain from (4.9), (4.10) the pointwise bound:

$$\frac{d}{dt}\left\{\int|\hat{w}_2|^2\right\}\lesssim\frac{b^3|\log b|^2}{\lambda^4},$$

which we integrate using also (2.75):

$$\mathcal{E}_2(t) = \lambda^2(t)\|w_2(t)\|^2_{L^2}\lesssim\|H\zeta_b(t)\|^2_{L^2}+\lambda^2(t)\|\hat{w}_2(t)\|^2_{L^2} \lesssim b^4(t)|\log b(t)|^2+\left(\frac{\lambda(t)}{\lambda(0)}\right)^2\left[\mathcal{E}_2(0)+b^2(0)|\log b(0)|^2\right]+\lambda^2(t)\int_0^t\frac{b^3|\log b|^2}{\lambda^4(\tau)}d\tau. \tag{4.12}$$

From (3.12), (4.4):

$$\left(\frac{\lambda(t)}{\lambda(0)}\right)^2\left[\mathcal{E}_2(0)+b^2(0)|\log b(0)|^2\right] \lesssim \frac{(b(0))^{10}+b^2(0)|\log b(0)|^2}{(b(0)|\log b(0)|^{\alpha_2}|)^2}b^2(t)|\log b(t)|^{2\alpha_2} \leq b^2(t)|\log b(t)|^{4+\frac{1}{4}},$$

We now use the bound $b_s\lesssim -b^2$ and (4.4) to estimate:

$$\begin{aligned}\lambda^2(t)\int_0^t\frac{b^3|\log b|^2}{\lambda^4(\tau)}d\tau&\lesssim\lambda^2(t)\int_0^t\frac{-b_tb|\log b|^2}{\lambda^2(\tau)}d\tau\\ &\lesssim \left(\frac{\lambda(t)}{\lambda(0)}\right)^2b^2(0)|\log b(0)|^{2\alpha_1}\int_0^t\frac{-b_t}{b|\log b|^{2\alpha_1-2}}d\tau\\ &\lesssim \left(\frac{\lambda(t)}{\lambda(0)}\right)^2b^2(0)|\log b(0)|^{2\alpha_1}\frac{1}{|\log b(0)|^{2\alpha_1-3}}\\ &\lesssim b^2(t)|\log b(t)|^{2\alpha_2}\frac{|\log b(0)|^3}{|\log b(0)|^{2\alpha_2}}\lesssim b^2(t)|\log b(t)|^{4+\frac{1}{4}}.\end{aligned}$$

Injecting these bounds into (4.12) yields:

$$\mathcal{E}_2(t)\lesssim b^2(t)|\log b(t)|^{4+\frac{1}{4}}$$

and concludes the proof of (3.19).

Proof of (4.11): We estimate the contribution of each term in (4.11) coming from the decomposition (4.8). First observe from the interpolation bound (B.5) and (2.75):

$$\int_{y\leq 2B_0}|\hat{\varepsilon}_2|^2 \lesssim B_0^4|\log b|^2\int\frac{|\varepsilon|^2}{(1+y^4)|\log y|^2}+\int|H\zeta_b|^2 \lesssim C(M)b^2+b^2|\log b|^2\lesssim b^2|\log b|^2. \tag{4.13}$$

The $\hat{\Psi}_b$ term is now estimated using (2.77) and (4.13):

$$|(\hat{\varepsilon}_2, H\hat{\Psi}_b)| \lesssim \|H\hat{\Psi}_b\|_{L^2}\|\hat{\varepsilon}_2\|_{L^2(y\leq 2B_0)} \lesssim \left(b^4|\mathrm{log}b|^2 b^2|\mathrm{log}b|^2\right)^{\frac12} \lesssim b^3|\mathrm{log}b|^2.$$

We next estimate from (2.31), (2.45):

$$\int |H\hat{T}_1|^2 \lesssim \int_{y\leq 2B_0}|\Lambda Q|^2 + \int_{B_0\leq y\leq 2B_0}\left|\frac{\mathrm{log}y}{y}\right|^2 \lesssim |\mathrm{log}b|^2,$$

$$\int |H\hat{T}_2|^2 \lesssim \int_{y\leq 2B_0}\left|\frac{y}{y^2 b|\mathrm{log}b|}\right|^2 \lesssim \frac{1}{b^2|\mathrm{log}b|},$$

and thus from (2.74), (3.28), (3.29):

$$\begin{aligned}\int |H\widehat{Mod}(t)|^2 &\lesssim \left|\frac{\lambda_s}{\lambda}+b\right|^2\int|H\Lambda\hat{Q}_b|^2 + |b_s+b^2|^2\int|H(\hat{T}_1+b\hat{T}_2)|^2\\ &\lesssim \frac{b^4}{|\mathrm{log}b|^2}|\mathrm{log}b|^2 \lesssim b^4|\mathrm{log}b|^2.\end{aligned}$$

Moreover, $\mathrm{Supp}(H\widehat{Mod}) \subset [0, 2B_0]$ and thus with (4.13):

$$|(\hat{\varepsilon}_2, H\widehat{Mod})| \lesssim \left(b^4|\mathrm{log}b|^2 b^2|\mathrm{log}b|^2\right)^{\frac12} \lesssim b^3|\mathrm{log}b|^2.$$

We now turn to the control of the small linear term $\hat{L}(\hat{\varepsilon})$ which we rewrite as for (3.50):

$$\hat{L}(\hat{\varepsilon}) = -\frac{N_2(\hat{\alpha})}{y^2}\hat{\varepsilon} \;\text{ with }\; N_2(\hat{\alpha}) = f'(Q+\hat{\alpha}) - f'(Q) = \hat{\alpha}\int_0^1 f''(Q+\tau\hat{\alpha})d\tau.$$

Near the origin $y \leq 1$, $\hat{\varepsilon} = \varepsilon$ and the high order vanishing (3.51) and the bounds (B.10), (B.11), (B.12) easily yield:

$$\int_{y\leq 1}|H\hat{L}(\hat{\varepsilon})|^2 \lesssim b^6.$$

For $y \geq 1$, we estimate like for (3.52):

$$|N_2(\hat{\alpha})| \lesssim b|\mathrm{log}b|^C \mathbf{1}_{y\leq 2B_0},$$

and then a brute force computation and (2.75) yield the control:

$$\int_{y\geq 1}|H\hat{L}(\hat{\varepsilon})|^2 \lesssim b^2|\mathrm{log}b|^C\int_{y\leq 2B_0}\Sigma_{i=0}^2\frac{|\partial_y^i\hat{\varepsilon}|^2}{y^{2(4-i)}} \lesssim b^2|\mathrm{log}b|^C b^4|\mathrm{log}b|^C \lesssim b^5.$$

We also estimate from (2.75) and the bootstrap bound (3.15):

$$\|\hat{\varepsilon}_2\|_{L^2}^2 \lesssim b^2|\mathrm{log}b|^C \tag{4.14}$$

and thus

$$|(\hat{\varepsilon}_2, H\hat{L}(\hat{\varepsilon}))| \lesssim \left(b^2|\mathrm{log}b|^C b^5\right)^{\frac12} \leq b^3|\mathrm{log}b|^2.$$

It remains to estimate the nonlinear term. Near the origin, we argue like for the proof of (3.53) to derive:

$$\forall y\leq 1, \quad |H(\hat{N}(\hat{\varepsilon}))| = |H(N(\varepsilon))| \lesssim b^4|\mathrm{log}b|^C,$$

this is left to the reader. For $y \geq 1$, we introduce the decomposition:

$$\hat{N}(\hat{\varepsilon}) = \hat{z}^2\hat{N}_0(\hat{\varepsilon}) \;\text{ with }\; \hat{z} = \frac{\hat{\varepsilon}}{y}, \quad \hat{N}_0(\hat{\varepsilon}) = \int_0^1(1-\tau)f''(\hat{Q}_b+\tau\hat{\varepsilon})d\tau,$$

and recall the formula (3.60):

$$H(\hat{N}(\varepsilon)) = \hat{N}_0(\hat{\varepsilon})H(\hat{z}^2) - 2\hat{z}\partial_y\hat{z}\partial_y\hat{N}_0(\hat{\varepsilon}) - \hat{z}^2\Delta(\hat{N}_0(\hat{\varepsilon})).$$

The estimates (3.56), (3.57), (3.58), (3.59) still hold from direct check for $\hat{N}_0(\hat{\varepsilon})$, and we estimate with Lemma B.3 and (2.76):

$$\begin{aligned}
\int_{y\geq 1}|\hat{N}_0(\hat{\varepsilon})H(\hat{z}^2)|^2 &\lesssim \int_{y\geq 1}\left[|\hat{z}\Delta\hat{z}|^2+\frac{\hat{z}^4}{y^4}+|\partial_y\hat{z}|^4\right]\\
&\lesssim \left[\left\|\frac{\hat{\varepsilon}}{y}\right\|^2_{L^\infty(y\geq 1)}+\|\partial_y\hat{\varepsilon}\|^2_{L^\infty(y\geq 1)}\right]\Sigma_{i=0}^2\int\frac{|\partial_y^i\hat{\varepsilon}|^2}{1+y^{2(3-i)}}\\
&\lesssim b^2|\text{log}b|^Cb^3|\text{log}b|^C\lesssim b^5|\text{log}b|^C.
\end{aligned}$$

$$\begin{aligned}
\int_{y\geq 1}|\hat{z}\partial_y\hat{z}\partial_y\hat{N}_0(\hat{\varepsilon})|^2 &\lesssim |\text{log}b|^C\left\|\frac{\hat{\varepsilon}}{y}\right\|^2_{L^\infty(y\geq 1)}\int_{y\geq 1}\left[\frac{1}{y^4}+b^2\right]\left[\frac{|\partial_y\hat{\varepsilon}|^2}{y^2}+\frac{|\hat{\varepsilon}|^2}{y^4}\right]\\
&\lesssim b^2|\text{log}b|^Cb^3|\text{log}b|^C\lesssim b^5|\text{log}b|^C,
\end{aligned}$$

$$\int|\hat{z}^2\Delta(\hat{N}_0(\hat{\varepsilon}))|^2\lesssim|\text{log}b|^C\left\|\frac{\hat{\varepsilon}}{y}\right\|^2_{L^\infty(y\geq 1)}\int_{y\geq 1}\left[\frac{1}{y^6}+b^3\right]\frac{|\hat{\varepsilon}|^2}{y^2}\lesssim b^5|\text{log}b|^C.$$

We thus conclude using (4.14):

$$|(\hat{\varepsilon}_2,H\hat{N}(\hat{\varepsilon}))|\lesssim\left(b^2|\text{log}b|^Cb^5\right)^{\frac{1}{2}}\leq b^3|\text{log}b|^2.$$

This concludes the proof of (4.11).
This concludes the proof of the Proposition 3.1.

4.2. **Proof of Theorem 1.1.** We are now in position to conclude the proof of Theorem 1.1. The proof relies on the reintegration of the modulation equations as in [15], [17], [19], we sketch the argument for the sake of completeness.

step 1 Finite time blow up.

Let $T\leq+\infty$ be the life time of the full map v given by (3.2), then the estimates of Proposition 3.1 hold on $[0,T)$. From (3.30), (4.4),

$$-\frac{d}{dt}\sqrt{\lambda}=-\frac{1}{2\lambda\sqrt{\lambda}}\frac{\lambda_s}{\lambda}\gtrsim\frac{b}{\lambda\sqrt{\lambda}}\gtrsim C(u_0)>0$$

and thus λ touches zero at some finite time $T_0<+\infty$. Using (4.27), (4.28), it is easily seen that the estimates of Proposition 3.1 and the bootstrap bounds of Proposition (B.3) imply:

$$\forall t\in[0,T_0),\quad \|\Delta v(t)\|_{L^2}\lesssim C(t)<+\infty$$

and thus from the blow up criterion (3.1):

$$T_0=T<+\infty.$$

Observe then from (4.4) that this implies

$$\lambda(T)=b(T)=0. \tag{4.15}$$

Moreover, we obtain from (3.28) and the estimates of Proposition 3.1 the rough bound:

$$|\lambda\lambda_t|=\left|\frac{\lambda_s}{\lambda}\right|\lesssim 1\ \text{ and thus }\ \lambda(t)\lesssim\sqrt{T-t},$$

and hence from (3.21):

$$s(t)\to+\infty\ \text{ as }\ t\to T. \tag{4.16}$$

step 2 Derivation of the sharp blow up speed.

We now slightly refine our control of b through a logarithmic gain in the modulation equation (3.29). We commute (3.22) with H and take the inner product with $\chi_{B_\delta}\Lambda Q$ to derive:

$$\begin{aligned}&\frac{d}{ds}\left\{(H\varepsilon,\chi_{B_\delta}\Lambda Q)\right\}-(H\varepsilon,\partial_s\chi_{B_\delta}\Lambda Q)+\frac{\lambda_s}{\lambda}(\chi_{B_\delta}\Lambda Q,H\Lambda\varepsilon)+(H^2\varepsilon,\chi_{B_\delta}\Lambda Q)\\ =&\left(H\left[-\tilde{\Psi}_b+L(\varepsilon)-N(\varepsilon)-Mod\right],\chi_{B_\delta}\Lambda Q\right).\end{aligned}\tag{4.17}$$

We now estimate all terms in the above identity. First, for δ small enough, we estimate in brute force:

$$|(H\varepsilon,\partial_s\chi_{B_\delta}\Lambda Q)|+|\frac{\lambda_s}{\lambda}(\chi_{B_\delta}\Lambda Q,H\Lambda\varepsilon)|+|(H[L(\varepsilon)-N(\varepsilon)],\chi_{B_\delta}\Lambda Q)|\lesssim\frac{b}{b^{C\delta}}\sqrt{\mathcal{E}_4}\lesssim\frac{b^2}{|\log b|^2}.$$

We then estimate the linear term:

$$|(H^2\varepsilon,\chi_{B_\delta}\Lambda Q)|\lesssim\sqrt{\mathcal{E}_4}\sqrt{|\log b|}\lesssim\frac{b^2}{\sqrt{|\log b|}}.$$

The leading order $\tilde{\Psi}_b$ term is computed from (2.57), (2.67):

$$(-H\tilde{\Psi}_b,\chi_{B_\delta}\Psi_b)=-b^2(H\Sigma_b,\chi_{B_\delta}\Psi_b)+O\left(\frac{b^3}{b^{C\delta}}\right)=b^2c_b(\Lambda Q,\chi_{B_\delta}\Lambda Q)+O\left(\frac{b^2}{|\log b|^2}\right).$$

Finally, we compute the modulation term from (2.60):

$$\begin{aligned}(-HMod,\chi_{B_\delta}\Lambda Q)&=\left(\frac{\lambda_s}{\lambda}+b\right)(H\Lambda\tilde{Q}_b,\chi_{B_\delta}\Lambda Q)-(b_s+b^2)(H(\tilde{T}_1+2b\tilde{T}_2),\chi_{B_\delta}\Lambda Q)\\&=(b_s+b^2)(\Lambda Q,\chi_{B_\delta}\Lambda Q)+O\left(\frac{b}{b^{C\delta}}\frac{b^2}{|\log b|}\right).\end{aligned}$$

We thus inject the collection of above estimates into (4.17) and derive the modulation equation:

$$(b_s+b^2)(\Lambda Q,\chi_{B_\delta}\Lambda Q)=\frac{d}{ds}\left\{(H\varepsilon,\chi_{B_\delta}\Lambda Q)\right\}-c_bb^2(\Lambda Q,\chi_{B_\delta}\Lambda Q)+O\left(\frac{b^2}{\sqrt{|\log b|}}\right)$$

which we rewrite using (2.32) and an integration by parts in time:

$$\begin{aligned}&\frac{d}{ds}\left\{b-\frac{(H\varepsilon,\chi_{B_\delta}\Lambda Q)}{(\Lambda Q,\chi_{B_\delta}\Lambda Q)}\right\}+b^2\left(1+\frac{2}{|\log b|}\right)\\ =&\ O\left(\frac{b^2}{|\log b|^{\frac{3}{2}}}\right)+(H\varepsilon,\chi_{B_\delta}\Lambda Q)\frac{(\Lambda Q,\partial_s\chi_{B_\delta}\Lambda Q)}{(\Lambda Q,\chi_{B_\delta}\Lambda Q)^2}.\end{aligned}\tag{4.18}$$

We now estimate:

$$\left|(H\varepsilon,\chi_{B_\delta}\Lambda Q)\frac{(\Lambda Q,\partial_s\chi_{B_\delta}\Lambda Q)}{(\Lambda Q,\chi_{B_\delta}\Lambda Q)^2}\right|\lesssim\frac{\sqrt{\mathcal{E}_4}}{b^{C\delta}}\frac{|b_s|}{b}\lesssim\frac{b^3}{b^{C\delta}},$$

$$\left|\frac{(H\varepsilon,\chi_{B_\delta}\Lambda Q)}{(\Lambda Q,\chi_{B_\delta}\Lambda Q)}\right|\lesssim\frac{\sqrt{\mathcal{E}_4}}{b^{C\delta}}\lesssim\frac{b^2}{b^{C\delta}}.$$

We inject these bounds into (4.18) and conclude that the quantity

$$\tilde{b}=b-\frac{(H\varepsilon,\chi_{B_\delta}\Lambda Q)}{(\Lambda Q,\chi_{B_\delta}\Lambda Q)}=b+O\left(\frac{b^2}{|\log b|^2}\right)\tag{4.19}$$

satisfies the pointwise differential control:

$$\left|\tilde{b}_s+\tilde{b}^2\left(1+\frac{2}{|\log\tilde{b}|}\right)\right|\lesssim\frac{\tilde{b}^2}{|\log\tilde{b}|^{\frac{3}{2}}}.$$

Equivalently,

$$\frac{\tilde{b}_s}{\tilde{b}^2\left(1+\frac{2}{|\log\tilde{b}|}\right)}+1=O\left(\frac{1}{|\log\tilde{b}|^{\frac{3}{2}}}\right).$$

We now integrate this in time using $\lim_{s\to+\infty}\tilde{b}(s)=0$ from (4.15), (4.16), (4.19) and get:

$$\tilde{b}(s)=\frac{1}{s}-\frac{2}{s\log s}+O\left(\frac{1}{s|\log s|^{\frac{3}{2}}}\right)$$

and thus from (4.19):

$$b(s)=\frac{1}{s}-\frac{2}{s\log s}+O\left(\frac{1}{s|\log s|^{\frac{3}{2}}}\right). \tag{4.20}$$

We now inject the modulation equation (3.2) and conclude:

$$-\frac{\lambda_s}{\lambda}=\frac{1}{s}-\frac{2}{s\log s}+O\left(\frac{1}{s|\log s|^{\frac{3}{2}}}\right).$$

We rewrite this as

$$\left|\frac{d}{ds}\log\left(\frac{s\lambda(s)}{(\log s)^2}\right)\right|\lesssim\frac{1}{s|\log s|^{\frac{3}{2}}}$$

and thus integrating in time yields the existence of $\kappa(u)>0$ such that:

$$\frac{s\lambda(s)}{(\log s)^2}=\frac{1}{\kappa(u)}\left[1+O\left(\frac{1}{|\log s|^{\frac{3}{2}}}\right)\right].$$

Taking the log yields the bound

$$|\log\lambda|=|\log s|\left[1+O\left(\frac{|\log\log s|}{\log s}\right)\right]$$

and thus

$$\frac{1}{s}=\kappa(u)\frac{\lambda}{|\log\lambda|^2}\left(1+o(1)\right).$$

Injecting this into (4.20) yields:

$$-\lambda\lambda_t=-\frac{\lambda_s}{\lambda}=\frac{1}{s}\left(1+o(1)\right)=\kappa(u)\frac{\lambda}{|\log\lambda|^2}\left(1+o(1)\right) \tag{4.21}$$

and thus

$$-|\log\lambda|^2\lambda_t=\kappa(u)(1+o(1)).$$

Integrating from t to T with $\lambda(T)=0$ yields

$$\lambda(t)=\kappa(u)\frac{T-t}{|\log(T-t)|^2}\left[1+o(1)\right],$$

and (1.13) is proved. This also implies using (3.30):

$$b(t)=\kappa^2(u)\frac{T-t}{|\log(T-t)|^4}\left[1+o(1)\right]. \tag{4.22}$$

In particular:

$$\frac{b}{\lambda}=\frac{k(u)}{|\log b|^2}(1+o(1)). \tag{4.23}$$

step 3 $\dot{H}^2$ bound.

We now turn to the proof of (1.12), (1.14). Let $v(t,x)$ be the map associated to $u(t,r)$, explicitly:

$$v(t,x) = \left| \begin{array}{l} g(u(t,r)\cos\theta \\ g(u(t,r))\sin\theta \\ z(u(t,r) \end{array} \right.$$

and $\mathcal{Q}$ be given by (1.11). Let then

$$\tilde{v}(t,x) = v(t,x) - \mathcal{Q}\left(\frac{x}{\lambda(t)}\right), \tag{4.24}$$

and correspondingly

$$\tilde{u}(t,r) = u(t,r) - Q\left(\frac{r}{\lambda(t)}\right) = (\tilde{\alpha} + \varepsilon)_{\lambda(t)}. \tag{4.25}$$

We claim the bound:

$$\forall t \in [0,T), \quad \|\Delta\tilde{v}(t,x)\|_{L^2} \leq C(v_0). \tag{4.26}$$

Indeed, let the normal vector to the revolution surface M at v be given by

$$\mathbf{n} = \left| \begin{array}{l} -z'(u)\cos\theta \\ -z'(u)\sin\theta \\ g'(u) \end{array} \right.$$

and compute the Laplace operator:

$$\Delta v = [\Delta v - (\Delta v \cdot \mathbf{n})\mathbf{n}] + (\Delta v \cdot \mathbf{n})\mathbf{n}$$

with explicitly:

$$\Delta v - (\Delta v \cdot \mathbf{n})\mathbf{n} = \left(\Delta u - \frac{f(u)}{r^2}\right) \left| \begin{array}{l} g'(u)\cos\theta \\ g'(u)\sin\theta \\ z'(u) \end{array} \right. , \tag{4.27}$$

$$\Delta v \cdot \mathbf{n}\mathbf{n} = \left| \begin{array}{l} \left[(\partial_r u)^2 g''(u) - \frac{g(u)(z'(u))^2}{r^2}\right]\cos\theta \\ \left[(\partial_r u)^2 g''(u) - \frac{g(u)(z'(u))^2}{r^2}\right]\sin\theta \\ (\partial_r u)^2 z''(u) - \frac{z(u)(z'(u))^2}{r^2} \end{array} \right. , \tag{4.28}$$

we now claim:

$$\int |\Delta\tilde{v} - (\Delta\tilde{v}\cdot\mathbf{n})\mathbf{n}|^2 \lesssim C(v_0), \tag{4.29}$$

$$\int |\Delta\tilde{v}\cdot\mathbf{n}|^2 \lesssim C(v_0), \tag{4.30}$$

which implies (4.26).

Proof of (4.29): We inject the decomposition (4.25) and estimate:

$$\int |\Delta\tilde{v} - (\Delta\tilde{v}\cdot\mathbf{n})\mathbf{n}|^2 = \int \left|\Delta u - \frac{f(u)}{r^2}\right|^2 = \frac{1}{\lambda^2}\int \left|\Delta(\tilde{\alpha}+\varepsilon) - \frac{f(Q+\tilde{\alpha}+\varepsilon) - f(Q)}{y^2}\right|^2$$

$$\lesssim \frac{1}{\lambda^2}\left[\int |H\varepsilon|^2 + \int |H\tilde{\alpha}|^2 + \int |f''(Q)|^2 \frac{|\tilde{\alpha}|^4 + |\varepsilon|^4}{y^4} + \int \frac{|\tilde{\alpha}|^6 + |\varepsilon|^6}{y^4}\right]. \tag{4.31}$$

We compute from the definition of $\tilde{\alpha}$ and (4.23):

$$\int |H\tilde{\alpha}|^2 \lesssim \int_{y\leq 2B_1} \left[by|\text{log}y| + b^3\frac{y^3}{b|\text{log}b|}\right|^2 \lesssim b^2|\text{log}b|^2 \lesssim \lambda^2,$$

$$\int \frac{|\tilde{\alpha}|^4 + |\tilde{\alpha}|^6}{y^4} \lesssim \int_{y\leq 2B_1} \frac{1}{y^4}\left[by|\text{log}y| + b^3\frac{y^3}{b|\text{log}b|}\right|^4 \lesssim b^3|\text{log}b|^C \lesssim \lambda^2.$$

Observe now from (4.12), (1.13), (4.22) the bound:

$$\int |H\varepsilon(t)|^2 = \mathcal{E}_2(t) \lesssim C(u_0)\lambda^2(t)\left[1+\int_0^T \frac{b^3(\tau)|\mathrm{log} b(\tau)|^2}{\lambda^4(\tau)}d\tau\right] \lesssim C(u_0)\lambda^2(t)$$

where we used from (1.13), (4.22) again:

$$\int_0^T \frac{b^3(\tau)|\mathrm{log} b(\tau)|^2}{\lambda^4(\tau)}d\tau \lesssim \int_0^T \frac{1}{(T-\tau)|\mathrm{log}(T-\tau)|^2}d\tau < +\infty.$$

We now estimate using (B.12):

$$\int_{y\leq 1} |f''(Q)|^2\frac{|\varepsilon|^4}{y^4} \lesssim b^6 \lesssim \lambda^2,$$

and using $|f''(Q)| \lesssim \frac{y}{1+y^2}$, (B.14) and the energy bound:

$$\int_{y\geq 1} |f''(Q)|^2\frac{|\varepsilon|^4}{y^4} \lesssim \left\|\frac{\varepsilon}{y^2}\right\|^2_{L^\infty(y\geq 1)}\int \frac{|\varepsilon|^2}{y^2} \lesssim b^3|\mathrm{log} b|^C \lesssim \lambda^2.$$

Injecting these bounds into (4.31) yields (4.29).
Proof of (4.30): We first claim:

$$\int \left|(\partial_r u)^2 g''(u) - (\partial_r Q_\lambda)^2 g''(Q_\lambda)\right|^2 \lesssim 1. \tag{4.32}$$

Indeed, we estimate:

$$\begin{aligned}\left|(\partial_r u)^2 g''(u) - (\partial_r Q_\lambda)^2 g''(Q_\lambda)\right| &\lesssim |(\partial_r u)^2(g''(u)-g''(Q_\lambda))| + |g''(Q_\lambda)||\partial_r u|^2 - |\partial_r Q_\lambda|^2| \\ &\lesssim |\tilde{u}|\left[|\partial_r Q_\lambda|^2 + |\partial_r \tilde{u}|^2\right] + |g''(Q_\lambda)|\left[|\partial_r Q_\lambda||\partial_r \tilde{u}| + |\partial_r \tilde{u}|^2\right]\end{aligned}$$

and thus after rescaling using Lemma B.3:

$$\begin{aligned}\lambda^2\int \left|(\partial_r u)^2 g''(u) - (\partial_r Q_\lambda)^2 g''(Q_\lambda)\right|^2 &\lesssim \int \frac{|\tilde{\alpha}|^2+|\varepsilon|^2}{1+y^8} + \int |\partial_y(\tilde{\alpha}+\varepsilon)|^4|\tilde{\alpha}+\varepsilon|^2 \\ &+ \int \frac{1}{1+y^6}|\partial_y(\tilde{\alpha}+\varepsilon)|^2 + \|\partial_y(\tilde{\alpha}+\varepsilon)\|^2_{L^\infty}\int \frac{1}{1+y^2}|\partial_y(\tilde{\alpha}+\varepsilon)|^2 \\ &\lesssim b^2 + \int (|\partial_y\tilde{\alpha}|^4 + |\partial_y\varepsilon|^4)(|\tilde{\alpha}|^2+|\varepsilon|^2).\end{aligned}$$

We estimate:

$$\int |\partial_y \alpha|^4|\tilde{\alpha}|^2 \lesssim b^4|\mathrm{log} b|^C, \quad \int |\partial_y\varepsilon|^4|\tilde{\alpha}|^2 \lesssim \|\partial_y\varepsilon\|^2_{L^\infty}\|\tilde{\alpha}\|^2_{L^\infty} \lesssim b^4|\mathrm{log} b|^C,$$

$$\int |\partial_y\tilde{\alpha}|^4|\varepsilon|^2 \lesssim b^4|\mathrm{log} b|^C\int_{y\leq 2B_1}|\varepsilon|^2 \lesssim b^4|\mathrm{log} b|^C B_1^8\int \frac{|\varepsilon|^2}{(1+y^8)|\mathrm{log} b|^C} \lesssim b^4|\mathrm{log} b|^C,$$

and using a two dimensional Gagliardo Nirenberg inequality for the radial function $\partial_y\varepsilon$:

$$\int |\partial_y\varepsilon|^4|\varepsilon|^2 \lesssim b^4 + \|\varepsilon\|^2_{L^\infty}\|\partial_{yy}\varepsilon\|^2_{L^2}\|\partial_y\varepsilon\|^2_{L^2} \lesssim b^4 + \mathcal{E}_2 \lesssim \lambda^2, \tag{4.33}$$

which concludes the proof of (4.32).
We now claim:

$$\int \left|\frac{g(u)(z'(u))^2 - g(Q_\lambda)(z'(Q_\lambda))^2}{r^2}\right|^2 \lesssim 1. \tag{4.34}$$

Indeed,

$$\left|\frac{g(u)(z'(u))^2 - g(Q_\lambda)(z'(Q_\lambda))^2}{r^2}\right| \lesssim \frac{|z'(Q_\lambda)|^2 + |\tilde{u}| + g(Q_\lambda)}{r^2}|\tilde{u}|$$

and then from $z'(0)=z'(\pi)=0$:

$$\int \frac{|z'(Q)|^4+|g(Q)|^2}{y^4}|\tilde{\alpha}+\varepsilon|^2 \lesssim \int \frac{|\tilde{\alpha}|^2+|\tilde{\varepsilon}|^2}{y^2(1+y^4)} \lesssim b^3|\log b|^C \lesssim \lambda^2,$$

$$\begin{aligned}\int \frac{|\tilde{u}|^2}{y^4}|\tilde{\alpha}+\varepsilon|^4 &\lesssim b^3|\log b|^C+\int_{y\geq 1}\frac{|\varepsilon|^4}{y^4}\\ &\lesssim b^3|\log b|^C+\int\left[|A\varepsilon|^4+|\partial_y\varepsilon|^4\right] \lesssim b^3|\log b|^C+\lambda^2+\int |\nabla(A\varepsilon)|^2\int |A\varepsilon|^2\\ &\lesssim \lambda^2+\int |A^*A\varepsilon|^2 \lesssim \lambda^2\end{aligned}$$

where we used (4.33), and (4.34) follows by rescaling. We estimate along the same lines:

$$\int \left|(\partial_r u)^2 z''(u)-(\partial_r Q_\lambda)^2 z''(Q_\lambda)-\frac{z(u)(z'(u))^2-z(Q_\lambda)(z'(Q_\lambda))^2}{r^2}\right|^2 \lesssim 1,$$

this is left to the reader. This concludes the proof of (4.30)

step 3 Quantization of the focused energy.

We now turn to the proof of (1.12), (1.14) and adapt the strategy in [18]. The regularity of $v(t,x)$ outside the origin is a standard consequence of parabolic regularity and the fact that in corotational symmetry the nonlinearity is singular at the origin only. Hence there exists $v^*\in \dot{H}^1$ such that

$$\forall R>0,\quad \nabla u(t)\to\nabla v^* \ \text{ in } \ L^2(|x|\geq R) \ \text{ as } \ t\to T.$$

Moreover, v is $\dot{H}^1$ bounded by conservation of energy, and thus recalling the decomposition (4.24) and the uniform bound (4.26):

$$\nabla\tilde{v}(t)\to\nabla v^* \ \text{ in } \ L^2 \ \text{ and } \ \Delta v^*\in L^2$$

which concludes the proof of (1.12), (1.14). This concludes the proof of Theorem 1.1.

Appendix A. Regularity in corotational symmetry

We detail in this appendix the regularity of $\dot{H}^1\cap\dot{H}^4$ maps with 1-corotational symmetry.

Lemma A.1 (Regularity in corotational symmetry)**.** *Let v be a 1-corotational map with*

$$\|\nabla v-\nabla \mathcal{Q}\|_{\dot{H}^1}\ll 1 \quad \text{and} \quad \|v\|_{\dot{H}^i}<+\infty,\quad 2\leq i\leq 4. \tag{A.1}$$

Then v admits a representation

$$v(x)=\left|\begin{array}{l} g(u(y))\cos\theta\\ g(u(y))\sin\theta\\ z(u(y))\end{array}\right. \quad \text{where} \quad u(y)=Q(y)+\varepsilon(y) \tag{A.2}$$

satisfies the boundary conditions

$$\varepsilon(0)=0,\quad \lim_{y\to+\infty}\varepsilon(y)=0, \tag{A.3}$$

the Sobolev bounds

$$\Sigma_{i=1}^4\int |\partial_y^{(i)}\varepsilon|^2+\int\frac{|\varepsilon|^2}{y^2}<+\infty \tag{A.4}$$

and the regularity at the origin:

$$\int |H^2\varepsilon|^2 + \int \frac{|H\varepsilon|^2}{y^4(1+|\mathrm{log} y|)^2} + \int \frac{|AH\varepsilon|^2}{y^2} + \int |\partial_y(AH\varepsilon)|^2 + \int \frac{|\partial_y^3\varepsilon|^2}{y^2(1+|\mathrm{log} y|)^2}$$
$$+ \int \frac{|\partial_y^2\varepsilon|^2}{y^4(1+|\mathrm{log} y|)^2} + \int \frac{|\partial_y\varepsilon|^2}{y^2(1+|\mathrm{log} y|)^2} + \int \frac{|\varepsilon|^2}{y^4(1+|\mathrm{log} y|)^2}$$
$$< +\infty. \tag{A.5}$$

Proof of Lemma A.1

step 1 $\dot{H}^1, \dot{H}^2$ bound.

From (A.2),

$$\int |\nabla v|^2 = 2\pi \int \left[|\partial_y u|^2 + \frac{g^2(u)}{y^2}\right]$$

and thus the structure of g, the Sobolev embedding in radial symmetry

$$\|u\|_{L^\infty}^2 \lesssim \|\partial_y u\|_{L^2} \|\frac{u}{y}\|_{L^2} \tag{A.6}$$

and the smallness (A.1) ensure:

$$\lim_{y\to 0} \varepsilon(y) = \lim_{y\to +\infty} \varepsilon(y) = 0.$$

The energy bound

$$\int |\partial_y \varepsilon|^2 + \frac{|\varepsilon|^2}{y^2} < +\infty \tag{A.7}$$

easily follows. Moreover, the energy density

$$e(y) = |\partial_1 v|^2 + |\partial_2 v|^2 = 2\left[|\partial_y u|^2 + \frac{g^2(u)}{y^2}\right]$$

is bounded near the origin from the regularity $v \in \dot{H}^1 \cap \dot{H}^4$ which implies

$$|\partial_y u| + \frac{|u(y)|}{y} \lesssim 1 \quad \text{for} \quad y \le 1. \tag{A.8}$$

We now recall (4.27) which implies using the $\dot{H}^2$ boundedness of v:

$$\int \left|\Delta u - \frac{f(u)}{y^2}\right|^2 < +\infty.$$

Using (A.7), (A.8) and the structure of f, we conclude:

$$\int |Hu|^2 < +\infty.$$

step 2 $\dot{H}^4$ bound.

We now recall (4.27), (4.28) which we rewrite:

$$\Delta v = H\varepsilon \left| \begin{array}{l} \cos\theta \\ \sin\theta \\ 0 \end{array} \right. + F(\varepsilon) \tag{A.9}$$

with:

$$F(\varepsilon) = \left(\Delta u - \frac{f(u)}{y^2}\right)\left|\begin{array}{l}(g'(u)-1)\cos\theta\\(g'(u)-1)\sin\theta\\z'(u)\end{array}\right. + \frac{f(Q+\varepsilon)-f(Q)-\varepsilon f'(Q)+(f'(Q)-1)\varepsilon}{y^2}\left|\begin{array}{l}\cos\theta\\\sin\theta\\0\end{array}\right.$$

$$+ \left|\begin{array}{l}\left[(\partial_r u)^2 g''(u) - \frac{g(u)(z'(u))^2}{r^2}\right]\cos\theta\\\left[(\partial_r u)^2 g''(u) - \frac{g(u)(z'(u))^2}{r^2}\right]\sin\theta\\(\partial_r u)^2 z''(u) - \frac{z(u)(z'(u))^2}{r^2}\end{array}\right. ,$$

We then compute $\nabla\Delta v$, $\Delta^2 v$ and use the odd parity of f, g and the cancellation at the origin (A.8) to conclude after a brute force computation:

$$\int |\partial_y H\varepsilon|^2 + \int \frac{|H\varepsilon|^2}{y^2} + \int |H^2\varepsilon|^2 < +\infty. \tag{A.10}$$

The Sobolev bound (A.4) away from the origin now easily follows:

$$\Sigma_{i=1}^4 \int_{y\geq 1} |\partial_y^{(i)}\varepsilon|^2 < +\infty.$$

step 3 Regularity at the origin.

We claim that (A.10) also implies the regularity (A.5) at the origin. Indeed, we estimate from (A.6) and the two dimensional Hardy inequality with logarithmic loss:

$$\begin{aligned}&\int_{y\leq 1}\left[\frac{|\partial_y H\varepsilon|^2}{y^2(1+|\log y|^2)} + \frac{|H\varepsilon|^2}{y^4(1+|\log y|^2)}\right]\\ \lesssim\ & 1 + \int \frac{|\nabla\Delta v|^2}{y^2(1+|\log y|^2)} \lesssim 1 + \int |\Delta^2 v|^2 + \int_{1\leq y\leq 2} |\nabla\Delta v|^2\\ \lesssim\ & 1\end{aligned} \tag{A.11}$$

We similarly compute from (4.27) after an explicit computation:

$$\int_{y\leq 1} \frac{(AH\varepsilon)^2}{y^2} \lesssim 1 + \int \left|\frac{1}{r}\partial_\theta\partial_1\left(\Delta v - (\Delta v\cdot\mathbf{n})\mathbf{n}\right)\right|^2 < +\infty.$$

We now observe from

$$\partial_y(\log(\Lambda\phi)) = \frac{Z}{y}$$

that for any function h:

$$A^* h = \partial_y h + \frac{1+Z}{y} h = \frac{1}{y\Lambda\phi}\partial_y(y\Lambda\phi h),$$

and thus using the a priori bound (A.8):

$$A\varepsilon(y) = \frac{1}{y\Lambda\phi(y)}\int_0^y \tau\Lambda\phi(\tau)H\varepsilon(\tau)d\tau.$$

We then estimate from Cauchy Schwarz and Fubbini:

$$\begin{aligned}&\int_{y\leq 1}\frac{|A\varepsilon|^2}{y^5(1+|\log y|^2)}dy \lesssim \int_{0\leq y\leq 1}\int_{0\leq\tau\leq y}\frac{y^5}{y^9(1+|\log y|^2)}|H\varepsilon(\tau)|^2 dy d\tau\\ \lesssim\ & \int_{0\leq\tau\leq 1}|H\varepsilon(\tau)|^2\left[\int_{\tau\leq y\leq 1}\frac{dy}{y^4(1+|\log y|^2)}\right]d\tau \lesssim \int_{\tau\leq 1}\frac{|H\varepsilon(\tau)|^2}{\tau^3(1+|\log\tau|^2)}d\tau\\ \lesssim\ & 1.\end{aligned} \tag{A.12}$$

We now rewrite near the origin:

$$H\varepsilon = -\partial_y^2\varepsilon + \frac{1}{y}\left(-\partial_y\varepsilon + \frac{\varepsilon}{y}\right) + \frac{V-1}{y^2}\varepsilon = -\partial_y^2\varepsilon + \frac{A\varepsilon}{y} + \frac{(V-1)+(1-Z)}{y^2}\varepsilon$$

which implies using (A.11), (A.12), (A.8):

$$\begin{aligned}\int_{y\leq 1}\frac{|\partial_y^2\varepsilon|^2}{y^4(1+|\log y|^2)} &\lesssim \int_{y\leq 1}\frac{|H\varepsilon|^2}{y^4(1+|\log y|^2)} + \int_{y\leq 1}\frac{|A\varepsilon|^2}{y^6(1+|\log y|^2)} + \int_{y\leq 1}\frac{|\varepsilon|^2}{y^4(1+|\log y|^2)}\\ &\lesssim 1,\end{aligned}$$

$$\begin{aligned}\int_{y\leq 1}\frac{|\partial_y^3\varepsilon|^2}{y^2(1+|\log y|^2)} &\lesssim \int_{y\leq 1}\frac{|\partial_y H\varepsilon|^2}{y^2(1+|\log y|^2)} + \int_{y\leq 1}\frac{|H\varepsilon|^2}{y^4(1+|\log y|^2)}\\ &+ \int_{y\leq 1}\frac{|\partial_y\varepsilon|^2}{y^2(1+|\log y|^2)} + \int_{y\leq 1}\frac{|\varepsilon|^2}{y^4(1+|\log y|^2)}\\ &\lesssim 1\end{aligned}$$

This concludes the proof of Lemma A.1.

Appendix B. Coercivity bounds and interpolation estimates

We recall in this section the coercivity bounds we use involving the operators $H, \tilde{H}$ and their iterate. Let us start with the coercivity of $\tilde{H}$:

Lemma B.1 (Coericivity of $\tilde{H}$)**.** *Let ε_3 with*

$$\int \frac{|\varepsilon_3|^2}{y^2} < +\infty,$$

then

$$\mathcal{E}_4 = (\tilde{H}\varepsilon_3, \varepsilon_3) = \|A^*\varepsilon_3\|_{L^2}^2 \geq c_0\left[\int |\partial_y\varepsilon_3|^2 + \int \frac{|\varepsilon_3|^2}{y^2(1+|\log y|^2)}\right] \tag{B.1}$$

for some universal constant $c_0 > 0$.

Proof of Lemma B.1. The proof is immediate in the case of $\mathbb{S}^2$ target thanks to the sign

$$\tilde{V} = \frac{4}{1+y^2} \geq 0$$

which yields (B.1) from standard two dimensional weigthed Hardy inequality -see [19] for more details-. For a general g however, this sign property is lost and the proof relies on a standard compactness argument and the explicit knowledge of the kernel of A^*.

We first claim the subcoercivity property:

$$\begin{aligned}(\tilde{H}\varepsilon_3, \varepsilon_3) &\geq c_0\left(\int |\partial_y\varepsilon_3|^2 + \int_{y\leq 1}\frac{|\varepsilon_3|^2}{y^2} + \int_{y\geq 1}\frac{\varepsilon_3^2}{y^2(1+y^2)} + \int \frac{|\varepsilon_3|^2}{y^2(1+|\log y|^2)}\right)\\ &- \frac{1}{c_0}\int\frac{|\varepsilon_3|^2}{1+y^4}\end{aligned} \tag{B.2}$$

for some universal constant $c_0 > 0$. Indeed, we estimate near the origin using the expansion (2.16):

$$\int_{y\leq 1}\frac{\tilde{V}}{y^2}|\varepsilon_3|^2 \geq c_0\int_{y\leq 1}\frac{|\varepsilon_3|^2}{y^2} - \frac{1}{c_0}\int_{y\leq 1}|\varepsilon_3|^2.$$

For $y\geq 1$, the logarithmic Hardy bound

$$\int\frac{|\varepsilon_3|^2}{y^2(1+|\text{log}y|^2)}\lesssim\int|\nabla\varepsilon_3|^2+\int_{1\leq y\leq 2}|\varepsilon_3|^2$$

and the degeracy from (2.16) $|\tilde{V}(y)|\lesssim\frac{1}{y^2}$ yield (B.2).
We now prove (B.1) and argue by contradiction. Let a sequence $\varepsilon_3^{(n)}$ with

$$\int|\partial_y\varepsilon_3^{(n)}|^2+\int_{y\leq 1}\frac{|\varepsilon_3^{(n)}|^2}{y^2}+\int_{y\geq 1}\frac{|\varepsilon_3^{(n)}|^2}{y^2(1+|\text{log}y|^2)}=1,\ \ (\tilde{H}\varepsilon_3^{(n)},\varepsilon_3^{(n)})\leq\frac{1}{n},\qquad\text{(B.3)}$$

then the sequence $\varepsilon_3^{(n)}$ is bounded in H^1_{loc} and thus weakly converges to ε_3^* up to a subsequence. Moreoever, (B.2),

$$\int\frac{|\varepsilon_3^{(n)}|^2}{1+y^4}\gtrsim c_0>0$$

and thus by lower semi continuity of norms and the compactness of the Sobolev embedding $H^1\hookrightarrow L^2_{loc}$:

$$\int|\partial_y\varepsilon_3^*|^2+\int_{y\leq 1}\frac{|\varepsilon_3^*|^2}{y^2}+\int_{y\geq 1}\frac{|\varepsilon_3^*|^2}{y^2(1+|\text{log}y|^2)}=1,\ \ \int\frac{|\varepsilon_3^*|^2}{1+y^4}\gtrsim c_0>0\qquad\text{(B.4)}$$

and

$$(\tilde{H}\varepsilon_3^*,\varepsilon_3^*)=\|A^*\varepsilon_3^*\|^2_{L^2}\leq 0.$$

Hence

$$A^*\varepsilon_3=0\ \text{ ie }\ \varepsilon_3=\frac{c}{y\Lambda Q}$$

which contradicts the boundary condition at the origin imposed by (B.4). This concludes the proof of (B.1) and Lemma B.1.

We now claim the following coercivity properties for H,H^2 which rely on a similar compactness argument and the explicit knowledge of the kernel of H,H^2. The proof is given in [22], [19] in the case $g(u)=\sin u$, but the same argument applies under our assumptions on g, the key being the behaviour at the origin and infinity (2.14), (2.15), (2.16). The proof is therefore left to the reader:

Lemma B.2 (Coercivity of H,H^2)**.** *Let $M>0$ and ε radially symmetric satisfying* (A.3)*,* (A.4)*,* (A.5) *and the orthogonality conditions*

$$(\varepsilon,\Phi_M)=(\varepsilon,H\Phi_M)=0.$$

Then there exists $C(M)>0$ such that the following bounds hold:

$$(H\varepsilon,\varepsilon)=\int|A\varepsilon|^2\geq C(M)\int\left[(\partial_y\varepsilon)^2+\frac{\varepsilon^2}{y^2}\right],$$

$$\int(H\varepsilon)^2\geq C(M)\int\left[\frac{(\partial_y\varepsilon)^2}{y^2(1+|\text{log}y|^2)}+\frac{\varepsilon^2}{y^4(1+|\text{log}y|^2)}\right],$$

$$\begin{aligned}\int(H^2\varepsilon)^2\ &\geq\ C(M)\int\left[\frac{|H\varepsilon|^2}{y^4(1+|\text{log}y|)^2}+\frac{|\partial_yH\varepsilon|^2}{y^2(1+|\text{log}y|)^2}+\int\frac{|\partial_y^3\varepsilon|^2}{y^2(1+|\text{log}y|)^2}\right.\\ &+\ \left.\frac{|\partial_y^2\varepsilon|^2}{y^4(1+|\text{log}y|)^2}+\int\frac{|\partial_y\varepsilon|^2}{y^2(1+y^4)(1+|\text{log}y|)^2}+\int\frac{|\varepsilon|^2}{y^4(1+y^4)(1+|\text{log}y|)^2}\right]\end{aligned}$$

We now recall the interpolation bounds on ε needed all along the proof of Proposition 3.1 which are a direct consequence of two dimensional weighted Hardy estimates and the coercivity bounds of Lemma B.2. These bounds which hold in the setting of the bootstrap bounds (3.13), (3.14), (3.15), (3.16) were explicitly derived in [19] to which we refer for a proof.

Lemma B.3 (Interpolation estimates)**.** *There holds -with constants a priori depending on M-:*

$$\int \frac{|\varepsilon|^2}{y^4(1+y^4)(1+|\mathrm{log}y|^2)} + \int \frac{|\partial_y^i \varepsilon|^2}{y^2(1+y^{6-2i})(1+|\mathrm{log}y|^2)} \lesssim \mathcal{E}_4, \quad 1 \le i \le 3, \tag{B.5}$$

$$\int_{y\geq 1} \frac{1+|\mathrm{log}y|^C}{y^2(1+|\mathrm{log}y|^2)(1+y^{6-2i})}|\partial_y^i\varepsilon|^2 \lesssim b^4|\mathrm{log}b|^{C_1(C)}, \quad 0\le i\le 3, \tag{B.6}$$

$$\int_{y\geq 1} \frac{1+|\mathrm{log}y|^C}{y^2(1+|\mathrm{log}y|^2)(1+y^{4-2i})}|\partial_y^i\varepsilon|^2 \lesssim b^3|\mathrm{log}b|^{C_1(C)}, \quad 0\le i\le 3, \tag{B.7}$$

$$\|\varepsilon\|_{L^\infty} \lesssim \delta(\alpha^*), \tag{B.8}$$

$$\|A\varepsilon\|^2_{L^\infty} \lesssim b^2|\mathrm{log}b|^2, \tag{B.9}$$

$$\left\|\frac{A\varepsilon}{y^2(1+|\mathrm{log}y|)}\right\|^2_{L^\infty(y\le 1)} + \left\|\frac{\Delta A\varepsilon}{1+|\mathrm{log}y|}\right\|^2_{L^\infty(y\le 1)} + \left\|\frac{H\varepsilon}{y(1+|\mathrm{log}y|)}\right\|^2_{L^\infty(y\le 1)} \lesssim b^4, \tag{B.10}$$

$$\left\|\frac{|H\varepsilon|}{y(1+|\mathrm{log}y|)}\right\|^2_{L^\infty(y\le 1)} \lesssim b^4, \tag{B.11}$$

$$\|\frac{\varepsilon}{y}\|^2_{L^\infty(y\le 1)} + \left\|\frac{\partial_y\varepsilon}{\sqrt{1+|\mathrm{log}y|}}\right\|^2_{L^\infty(y\le 1)} \lesssim b^4, \tag{B.12}$$

$$\|\frac{\varepsilon}{y}\|^2_{L^\infty(y\ge 1)} + \|\partial_y\varepsilon\|^2_{L^\infty(y\ge 1)} \lesssim b^2|\mathrm{log}b|^C, \tag{B.13}$$

$$\|\frac{\varepsilon}{1+y^2}\|^2_{L^\infty} + \|\frac{\partial_y\varepsilon}{1+y}\|^2_{L^\infty} + \|\partial_{yy}\varepsilon\|^2_{L^\infty(y\ge 1)} \lesssim b^3|\mathrm{log}b|^C, \tag{B.14}$$

$$\|\partial_{yyy}\varepsilon\|^2_{L^\infty(y\ge 1)} \lesssim b^4|\mathrm{log}b|^C. \tag{B.15}$$

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