TYPE II BLOW-UP FOR THE FOUR DIMENSIONAL ENERGY CRITICAL SEMI LINEAR HEAT EQUATION

RÉMI SCHWEYER

Abstract. We consider the energy critical four dimensional semi linear heat equation $\partial_t u - \Delta u - u^3 = 0$. We show the existence of type II finite time blow-up solutions and give a sharp description of the corresponding singularity formation. These solutions concentrate a universal bubble of energy in the critical topology

$$u(t,r) - \frac{1}{\lambda} Q\left(\frac{r}{\lambda(t)}\right) \to u^* \quad \text{in} \quad \dot{H}^1$$

where the blow-up profile is given by the Talenti Aubin soliton

$$Q(r) = \frac{1}{1+\frac{r^2}{8}},$$

and with speed

$$\lambda(t) \sim \frac{T-t}{|\log(T-t)|^2} \quad \text{as} \quad t \to T.$$

Our approach uses a robust energy method approach developed for the study of geometrical dispersive problems(Raphaël and Rodnianski, 2012 [18], Merle et al. 2011 [16]), and lies in the continuation of the study of the energy critical harmonic heat flow (Raphaël and Schweyer, 2011[19]) and the energy critical four dimensional wave equation (Hillaret and Raphaël, 2010[5]).

1. Introduction

1.1. Setting of the problem.

We consider in this paper the energy critical semi linear heat equation

$$\partial_t u - \Delta u - u^3 = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^4 \tag{1.1}$$

which is the energy critical four dimensional version of the more general problem

$$\partial_t u - \Delta u - u^p = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \quad p \geq 2^* - 1 \tag{1.2}$$

where

$$2^* = \frac{2N}{N-2}$$

is the Sobolev exponent. There is an important literature devoted to the qualitative description of solutions to (1.2), and we refer to [8], [9] for a complete introduction to the history of the problem. For radial data, two type of blow-up regimes are typically expected: type I blow-up which corresponds to a self-similar blow-up, and type II blow-up which displays excited blow-up speeds. Such kind of type II blow-up solutions were formally predicted by Herrero and Velázquez [4] using matching asymptotic procedures for a large value of p, and the corresponding regime displays a polynomial type blow-up speed. A major breakthrough is achieved by Matano and Merle in [8], [9], where the *non-existence of type II* blow-up is shown for

$$2^* - 1 < p < p_c$$

where

$$p_c = \left\{ \begin{array}{l} +\infty \quad \text{for} \quad N \leq 10 \\ 1 + \frac{4}{N-4-2\sqrt{N-1}} \quad \text{for} \quad N \geq 11 \end{array} \right. ,$$

and the *existence* of type II blow-up for $p > p_c$ is proved. More precisely, such solutions are obtained as *threshold* dynamics between well known type I blow-up solutions and global dissipative dynamics. A complete classification of these type II regimes is then completed in [10] where quantized blow-up speeds are exhibited with polynomial rates.

These results leave completely open the question of existence of type II blow-up in the energy critical setting. In fact, in the energy critical setting and even for the parabolic problem, the maximum principle does not seem to yield enough information to control a type II blow-up. The criticality of the problem is reflected by the fact that the total dissipated energy

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u(t,x)|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} u^{p+1}(t,x)dx \tag{1.3}$$

is left invariant by the scaling symmetry of the problem

$$u_\lambda(t,x) = \lambda^{\frac{N-2}{2}} u\left(\lambda^2 t, \lambda x\right), \quad E(u_\lambda(t)) = E(u(\lambda^2 t)).$$

The study of *critical problems* has attracted a considerable attention for the past ten years in the dispersive community, in particular the study of the mass critical nonlinear Schrödinger equation [17], [11], [12], [13], [14], [15] and geometric problems like wave maps, Schrödinger maps and the harmonic heat flow [6], [20], [18], [16], [19]. In particular, a robust *energy approach* is developed in [18], [16] to construct type II blow-up solutions in the energy critical setting. This strategy is implemented in the parabolic setting in [19] and led to the construction of *stable* blow-up dynamics with sharp asymptotics on the singularity formation for the harmonic heat flow. Note that more type II regimes for dispersive problems were obtained in [6], [7] but rely on the construction of *non-smooth* solutions and a procedure of backwards in time integration of the flow from the singularity which are both non suitable for parabolic problems.

1.2. **Statement of the result.** We carry out in this paper the program which was implemented in [5] to adapt the study of the geometric wave equation in [18] to the semi linear cubic four dimensional wave equation. The main difficulty is the fact that the energy (1.3) is non-definite positive, and this induces a non-positive eigenvalue in the spectrum of the linearized operator close to the Talenti-Aubin stationary solution

$$Q(r) = \frac{1}{1+\frac{r^2}{8}}, \tag{1.4}$$

which is the unique up to scaling radially symmetric solution to the stationary problem

$$\Delta Q + Q^3 = 0, \quad Q(r) = \frac{1}{1+\frac{r^2}{8}}. \tag{1.5}$$

This requires building our set of initial data on a suitable codimension one set, and we similarly claim in the continuation of [19] the existence of a type II blow-up dynamics for the energy critical four dimensional problem:

Theorem 1.1 (Existence of type II blow-up in dimension $N = 4$)**.** *Let Q be the Talenti-Aubin soliton* (1.5)*. Then $\forall \alpha^* > 0$, there exists a radially symmetric initial* *data* $u_0 \in H^1(\mathbb{R}^4)$ *with*

$$E(Q) < E(u_0) < E(Q) + \alpha^* \tag{1.6}$$

such that the corresponding solution to the energy critical focusing parabolic equation (1.1) *blows up in finite time* $T = T(u_0) < \infty$ *in a type II regime according to the following dynamics: there exists* $u^* \in \dot{H}^1$ *such that:*

$$\nabla \left[u(t,x) - \frac{1}{\lambda(t)} Q\left(\frac{x}{\lambda(t)}\right)\right] \to \nabla u^* \quad \text{in} \quad L^2 \quad \text{as} \quad t \to T \tag{1.7}$$

at the speed

$$\lambda(t) = c(u_0)\,(1 + o(1))\,\frac{T-t}{|\log(T-t)|^2} \quad \text{as} \quad t \to T \tag{1.8}$$

for some $c(u_0) > 0$*. Moreover, there hold the regularity of the asymptotic profile:*

$$\Delta u^* \in L^2 \tag{1.9}$$

Comments on the result

1. In dimension four, the choice $p = 2^* - 1 = 3$ is therefore the only one for which a type II blow-up occurs for radial data. We have decided to focus onto the four dimensional case for the case of simplicity but the construction we propose could be addressed in a much more general setting. Let us insist also that it does not rely on the maximum principle and may therefore be addressed in the non-radial setting and for more complicated systems.

2. The blow-up speed (1.8) is also the one obtained for the energy critical harmonic heat flow in [19]. Following the heuristics developed in [1], we conjecture the existence of a sequence of *quantized* blow-up speeds with polynomial rates corrected by suitable logarithmic factors, and (1.8) is the *fundamental* which corresponds from the proof to a codimension one in some weak sense manifold of initial data.

The main open problem after this work is to obtain a complete classification of type II blow-up for the energy critical problem, both in the radially symmetric case and the non-symmetric case.

Aknowledgments This work is part of the author's PhD thesis which is supported by the ERC/ANR project SWAP. The author would like to thank his advisor Pierre Raphaël for his guidance and advice.

Notations We introduce the differential operator

$$\Lambda f = f + y \cdot \nabla f \quad \text{(energy critical scaling)}.$$

Given a positive number $b > 0$, we let

$$B_0 = \frac{1}{\sqrt{b}}, \quad B_1 = \frac{|\log b|}{\sqrt{b}}. \tag{1.10}$$

Given a parameter $\lambda > 0$, we let

$$u_\lambda(r) = \frac{1}{\lambda} u(y) \quad \text{with} \quad y = \frac{r}{\lambda}.$$

We let χ as a smooth cut-off function with

$$\chi(y) = \begin{cases} 1 & \text{for} \quad y \le 1, \\ 0 & \text{for} \quad y \ge 2. \end{cases}$$

We shall systematically omit the measure in all radial two dimensional integrals and note:

$$\int f = \int_0^{+\infty} f(r) r^3 dr.$$

2. **Construction of an explicit approximate solution**

The aim of this section is to construct an approximate blow-up solution of (1.1), which is close to the ground state Q. This approximate solution will be the dominant part of the blow-up profile inside the parabolic zone. We adapt the strategy developed in [16], [18], [19].

Let u be a stationary solution of (1.1). Let $\lambda > 0$. Then, $\frac{1}{\lambda} u(\frac{r}{\lambda})$ is also a solution. Consider now that λ is no more a constant, but depends of time. Thus, we obtain the following equation:

$$\lambda(t)^2 \partial_t u\left(\frac{r}{\lambda(t)}\right) - \lambda(t)\lambda'(t)\Lambda u\left(\frac{r}{\lambda(t)}\right) - \Delta u\left(\frac{r}{\lambda(t)}\right) - u^3\left(\frac{r}{\lambda(t)}\right) = 0. \quad (2.1)$$

We then define a rescaled time

$$s = \int_0^t \frac{d\tau}{\lambda^2(\tau)}. \quad (2.2)$$

Remark that if $\lambda(t)$ verifies the law (1.8) defined in the Theorem 1.1, then $s(t)$ is a bijection between $[0, T[$ and $\mathbb{R}^+$. We also let the rescaled variable $y(t) = \frac{r}{\lambda(t)}$. Eq. (2.1) becomes, using the new variables:

$$\partial_s u - \frac{\lambda_s}{\lambda}\Lambda u - \Delta u - u^3 = 0. \quad (2.3)$$

As well as the parameter $\lambda(s)$, we define a new parameter $b(s)$ such that:

$$b = -\frac{\lambda_s}{\lambda}\left(1 + o(1)\right), \quad (2.4)$$

$$b_s = -b^2\left(1 + o(1)\right). \quad (2.5)$$

The modulation laws (2.4) and (2.5) will be justified thereafter. First, in the following subsection, we consider that

$$b_s = -b^2 \quad \text{and} \quad b + \frac{\lambda_s}{\lambda} = 0, \quad (2.6)$$

b being positive.

2.1. **Construction of explicit approximate blow-up profiles.**

Proposition 2.1 (Construction of the approximate profile)**.** *Let $M > 0$ be large enough. Then, there exists a small enough universal constant $b^*(M)$, such that the following holds. Let $b \in]0, b^*(M)[$. Then there exists profiles T_1, T_2 and T_3, such that*

$$Q_b(y) = Q(y) + bT_1(y) + b^2T_2(y) + b^3T_3(y) = Q(y) + \alpha(y)$$

generates an error

$$\Psi_b = -b^2(T_1 + 2bT_2) - \Delta Q_b - (Q_b)^3 + b\Lambda Q_b \quad (2.7)$$

which satisfies:
(i) Weighted bounds:

$$\int_{y \le 2B_1} |H\Psi_b|^2 \lesssim b^4 |\log b|^2, \quad (2.8)$$

$$\int_{y\leq 2B_1}\frac{1}{1+y^8}|\Psi_b|^2\lesssim b^6, \tag{2.9}$$

$$\int_{y\leq 2B_1}|H^2\Psi_b|^2\lesssim\frac{b^6}{|\log b|^2}. \tag{2.10}$$

(ii) Flux computation: Let Φ_M be given by (3.2), then:

$$\frac{(H\Psi_b,\Phi_M)}{(\Lambda Q,\Phi_M)}=-\frac{2b^2}{|\log b|}+O\left(\frac{b^2}{|\log b|^2}\right). \tag{2.11}$$

Remark 2.2. From the proof, the profiles $(T_i)_{1\leq i\leq 3}$ display a lower order dependance in b.

Proof of Proposition 2.1

Step1. Computation of the error.

We expand Q_b^3 and formulate the error Ψ_b as a polynomial expression in b:

$$\begin{aligned}Q_b^3 &= Q^3+3bQ^2T_1+b^2\left(3Q^2T_2+3QT_1^2\right)+b^3\left(3Q^2T_3+6QT_1T_2+T_1^3\right)\\ &+ R_1(T_1,T_2,T_3),\end{aligned}$$

where $R_1(T_1,T_2,T_3)$ is polynomial in $(T_i)_{1\leq i\leq 3}$ and contains the terms of power $(b^j)_{j\geq 4}$. Hence,

$$\begin{aligned}\Psi_b &= b\,(HT_1+\Lambda Q)\\ &+ b^2\left(HT_2-T_1+\Lambda T_1-3QT_1^2\right)\\ &+ b^3\left(HT_3-2T_2+\Lambda T_2-6QT_1T_2-T_1^3\right)\\ &+ b^4\Lambda T_3+R_1(T_1,T_2,T_3)\end{aligned} \tag{2.12}$$

with

$$H=-\Delta-3Q^2=-\Delta-V. \tag{2.13}$$

Morever,

$$V(y)=\frac{3}{\left(1+\frac{y^2}{8}\right)^2}, \tag{2.14}$$

which yields

$$V(y)=\begin{cases}3+O\left(y^2\right) & \text{as } y\to 0,\\ \dfrac{192}{y^4}+O\left(\dfrac{1}{y^6}\right) & \text{as } y\to+\infty,\end{cases} \tag{2.15}$$

and

$$\Lambda V=\frac{-192(3y^2-8)}{(y^2+8)^3}. \tag{2.16}$$

Step 2. Construction of T_1.

The spectral structure of the Schrödinger operator H is well known: it has a well localized non positive eigenvalue

$$H\psi=-\zeta\psi,\quad \zeta>0,$$

and a resonance at the origin induced by the energy critical scaling symmetry:

$$H\Lambda Q=0,\quad \Lambda Q\notin L^2(\mathbb{R}^4).$$

Hence the Green's functions of H are explicit and the other solution to $H\Gamma = 0$ for $y > 0$ is given by:

$$\Gamma(y) = -\Lambda Q \int_1^y \frac{dx}{x^3(\Lambda Q(x))^2} = \frac{y^2-8}{(y^2+8)^2}\left(\frac{y^2}{16} + 6\text{log}y - \frac{583}{112} - \frac{4}{y^2}\right) - \frac{64}{(y^2+8)^2},$$

which yields

$$\Gamma(y) = \begin{cases} O\left(\frac{1}{y^2}\right) & \text{as } \ y \to 0, \\ \frac{1}{16} + O\left(\frac{\text{log}y}{y^2}\right) & \text{as } \ y \to +\infty. \end{cases} \tag{2.17}$$

We may thus invert H explicitly and the smooth solutions at the origin of

$$Hu = f$$

are given by

$$u = \Gamma(y)\int_0^y f\Lambda Q - \Lambda Q(y)\int_0^y f\Gamma + c\Lambda Q(y), \quad c \in \mathbb{R}. \tag{2.18}$$

We let T_1 be the solution of

$$HT_1 + \Lambda Q = 0, \tag{2.19}$$

canceled in zero, which means that we choose the constant $c = 0$ in (2.18). There hold the behaviors at $r \to +\infty$

$$\Lambda^i T_1(y) = -4\left(\text{log}y - \frac{1}{2} + i\right) + O\left(\frac{(\text{log}y)^2}{y^2}\right), \quad \text{for } \ 0 \le i \le 2. \tag{2.20}$$

There hold the behaviors at $r \to 0$

$$\Lambda^i T_1 = O\left(y^2\right), \quad \text{for } \ 0 \le i \le 2. \tag{2.21}$$

Hence, for $0 \le i \le 2$

$$\begin{aligned} \|\Lambda^i T_1\|_{L^\infty_{y \le 2B_1}} &\lesssim \text{log}b, \\ H\Lambda^i T_1 &\sim \frac{8}{y^2}, \ \text{when } y \to \infty. \end{aligned}$$

Step 3. Construction of the radiation Σ_b.

First, we can notice that the choice $b_s = -b^2$ allows to cancel the log$y$ growth of the expression $-T_1 + \Lambda T_1 - 3QT_1^2$. We now construct a radiation term according two specifications. First, it must compensate the 1-growth of the last expression. Moreover the error induced by this term inside the parabolic zone $y \lesssim B_0$ must be sufficiently small in order not to perturb the dynamics of the blow-up.

Let

$$c_b = \frac{64}{\int \chi_{\frac{B_0}{4}}(\Lambda Q)^2} = \frac{2}{|\text{log}b|}\left(1 + O\left(\frac{1}{|\text{log}b|}\right)\right) \tag{2.22}$$

and

$$d_b = c_b\int_0^{B_0} \chi_{\frac{B_0}{4}}\Lambda Q\Gamma = O\left(\frac{1}{b|\text{log}b|}\right). \tag{2.23}$$

Let Σ_b be the solution to

$$H\Sigma_b = c_b\chi_{\frac{B_0}{4}}\Lambda Q + d_bH[(1-\chi_{3B_0})\Lambda Q] \tag{2.24}$$

given by

$$\Sigma_b(y) = \Gamma(y)\int_0^y c_b\chi_{\frac{B_0}{4}}(\Lambda Q)^2 - \Lambda Q(y)\int_0^y c_b\chi_{\frac{B_0}{4}}\Gamma\Lambda Q + d_b(1-\chi_{3B_0})\Lambda Q.$$

The choice of the constants c_b and d_b yields:

$$\Sigma_b = \begin{cases} c_b T_1 & \text{for } y \le \frac{B_0}{4} \\ 64\Gamma & \text{for } y \ge 6B_0 \end{cases} \qquad \Lambda\Sigma_b = \begin{cases} c_b \Lambda T_1 & \text{for } y \le \frac{B_0}{4} \\ 64\Lambda\Gamma & \text{for } y \ge 6B_0. \end{cases} \tag{2.25}$$

Then the estimates for Σ_b and $\Lambda\Sigma_b$ for $6B_0 \le y \le 2B_1$,

$$\Sigma_b(y) = 4 + O\left(\frac{\log y}{y^2}\right) \quad \Lambda\Sigma_b(y) = 4 + O\left(\frac{\log y}{y^2}\right), \tag{2.26}$$

which fits the first criterion, that we fixed to construct the radiation. For $\frac{B_0}{4} \le y \le 6B_0$, we have:

$$\begin{aligned} \Sigma_b(y) &= c_b\left(\frac{1}{16} + O\left(\frac{\log y}{y^2}\right)\right)\left[\int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2\right] - c_b\Lambda Q(y)\int_1^y O(x)dx \\ &= 4\frac{\int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2}{\int \chi_{\frac{B_0}{4}}(\Lambda Q)^2} + O\left(\frac{1}{|\log b|}\right). \end{aligned} \tag{2.27}$$

$$\begin{aligned} \Lambda\Sigma_b(y) &= \Lambda\Gamma(y)\int_0^y c_b\chi_{\frac{B_0}{4}}(\Lambda Q)^2 - \Lambda^2 Q(y)\int_0^y c_b\chi_{\frac{B_0}{4}}\Gamma\Lambda Q \\ &= c_b\left(\frac{1}{16} + O\left(\frac{\log y}{y^2}\right)\right)\left[\int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2\right] - c_b\Lambda^2 Q(y)\int_1^y O(x)dx \\ &= 4\frac{\int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2}{\int \chi_{\frac{B_0}{4}}(\Lambda Q)^2} + O\left(\frac{1}{|\log b|}\right). \end{aligned} \tag{2.28}$$

Similarly ,

$$\Lambda^2\Sigma_b(y) = 4\frac{\int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2}{\int \chi_{\frac{B_0}{4}}(\Lambda Q)^2} + O\left(\frac{1}{|\log b|}\right).$$

Eq. (2.24) and the cancellation $H\Lambda Q = 0$ yield the bounds:

$$\int |H\Sigma_b|^2 \lesssim \frac{1}{|\log b|}, \quad \int \frac{1}{1+y^8}|\Sigma_b|^2 \lesssim b^2, \quad \int |H^2\Sigma_b|^2 \lesssim \frac{b^2}{|\log b|^2}. \tag{2.29}$$

We will see that this bounds with respect to the second criterion for the construction of the radiation. Furthermore we will see the importance of the term c_b, which modifies the modulation equation of b. It becomes:

$$b_s = -b^2\left(1 + \frac{2}{|\log b|}\right). \tag{2.30}$$

After reintegration, this equation gives the expected blow-up speed (1.8).

Step 4. Construction of T_2.
Define

$$\Sigma_2 = \Sigma_b + T_1 - \Lambda T_1 - 3QT_1^2. \tag{2.31}$$

The profile T_2 will be defined later as the suitable output of H for the argument Σ_2. Estimate Σ_2 before choosingT_2. For $y \le 1$,

$$\Sigma_2 \lesssim y^2. \tag{2.32}$$

For $1 \leq y \leq 6B_0$

$$\begin{aligned}\Sigma_2 &= 4\left(\frac{\int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2}{\int \chi_{\frac{B_0}{4}}(\Lambda Q)^2} - 1\right) + O\left(\frac{(\log y)^2}{y^2}\right) + O\left(\frac{1}{|\log b|}\right) \\ &\lesssim \frac{1+\log(y\sqrt{b})}{|\log b|}. \end{aligned} \tag{2.33}$$

According to the choice of the modulation law for b and the construction of the radiation, we have, for $y \geq 6B_0$

$$\Sigma_2 \lesssim \frac{(\log y)^2}{y^2}. \tag{2.34}$$

Hence, we obtain:

$$|\Sigma_2| \lesssim \frac{y^2}{1+y^2}\left(\mathbf{1}_{y\leq 1} + \frac{1+\log(y\sqrt{b})}{|\log b|}\mathbf{1}_{1\leq y\leq 6B_0}\right) + \frac{(\log y)^2}{y^2}\mathbf{1}_{y\geq 6B_0}. \tag{2.35}$$

We have the same bound for $\Lambda\Sigma_2$ and for $\Lambda^2\Sigma_2$. We now let T_2 be the solution to

$$HT_2 = \Sigma_2 \tag{2.36}$$

given by

$$T_2(y) = \Gamma(y)\int_0^y \Sigma_2\Lambda Q - \Lambda Q(y)\int_0^y \Sigma_2\Gamma. \tag{2.37}$$

We derive from (2.35) the bounds:

$$\begin{aligned}\forall y \leq 2B_1, \quad |\Lambda^i T_2(y)| &\lesssim \frac{y^4}{1+y^4}\left(\mathbf{1}_{y\leq 1} + \frac{1}{b|\log b|}\mathbf{1}_{y\geq 1}\right), \quad 0 \leq i \leq 1 \quad (2.38)\\ |T_2(y)| &\lesssim y^2. \quad (2.39)\end{aligned}$$

With an explicit calculus, we prove that for any function f:

$$H\Lambda f = 2Hf + \Lambda Hf - \Lambda Vf. \tag{2.40}$$

Hence,

$$H(\Lambda T_2) = 2\Sigma_2 + \Lambda\Sigma_2 - \Lambda VT_2 \tag{2.41}$$

and

$$|H(\Lambda T_2)| \lesssim \frac{y^2}{1+y^2}\left(\mathbf{1}_{y\leq 1} + \frac{1+\log(y\sqrt{b})}{|\log b|}\mathbf{1}_{1\leq y\leq 6B_0}\right) + \frac{(\log y)^2}{y^2}\mathbf{1}_{y\geq 6B_0}. \tag{2.42}$$

Step 5. Construction of T_3.
In the same way as before, we define

$$\Sigma_3 = -2T_2 + \Lambda T_2 - 6QT_1T_2 - T_1^3. \tag{2.43}$$

Notice that we do not need to construct a second radiation term.We estimate from (2.20) and (2.38)

$$\forall y \leq 2B_1, \quad |\Sigma_3(y)| \lesssim \frac{y^4}{1+y^4}\left(\mathbf{1}_{y\leq 1} + \frac{1}{b|\log b|}\mathbf{1}_{y\geq 1}\right) \tag{2.44}$$

and

$$\forall y \leq 2B_1, \quad |\Lambda\Sigma_3(y)| \lesssim \frac{y^4}{1+y^4}\left(\mathbf{1}_{y\leq 1} + \frac{1}{b|\log b|}\mathbf{1}_{y\geq 1}\right). \tag{2.45}$$

We then let T_3 be the solution to

$$HT_3 = \Sigma_3 \tag{2.46}$$

given by:

$$T_3(y) = \Gamma(y)\int_0^y \Sigma_3 \Lambda Q - \Lambda Q(y)\int_0^y \Sigma_3\Gamma. \tag{2.47}$$

Hence,

$$\Lambda T_3(y) = \Lambda\Gamma(y)\int_0^y \Sigma_3 \Lambda Q - \Lambda^2 Q(y)\int_0^y \Sigma_3\Gamma. \tag{2.48}$$

We estimate from (2.44)

$$\forall y \leq 2B_1, \ \ |\Lambda^i T_3(y)| \lesssim \frac{y^6}{1+y^4}\left(\mathbf{1}_{y\leq 1} + \frac{1}{b|\log b|}\mathbf{1}_{y\geq 1}\right), \ \ 0 \leq i \leq 1 \tag{2.49}$$

$$|T_3(y)| \lesssim y^2(1+y^2). \tag{2.50}$$

Finally with (2.40),

$$H(\Lambda T_3) = 2\Sigma_3 + \Lambda\Sigma_3 - \Lambda V T_3 \tag{2.51}$$

and

$$\forall y \leq 2B_1, \ \ |H\Lambda T_3(y)| \lesssim \frac{y^4}{1+y^4}\left(\mathbf{1}_{y\leq 1} + \frac{1}{b|\log b|}\mathbf{1}_{y\geq 1}\right). \tag{2.52}$$

We have thus the bounds for $i = 0, 1$, using (2.52), (2.49), (2.44) and (2.45):

$$\int_{y\leq 2B_1} |H\Lambda^i T_3|^2 \lesssim \int_{y\leq 2B_1} \frac{1}{b^2|\log b|^2} \lesssim \frac{B_1^4}{b^2|\log b|^2} \lesssim \frac{|\log b|^2}{b^4}, \tag{2.53}$$

$$\int_{y\leq 2B_1} \frac{1}{1+y^8}|\Lambda^i T_3|^2 \lesssim \frac{1}{b^2|\log b|^2}\int_{y\leq 2B_1}\frac{1}{1+y^4} \lesssim \frac{1}{b^2}. \tag{2.54}$$

The crucial bound to control the error at H^4 level is:

$$\int_{y\leq 2B_1} |H^2\Lambda^i T_3|^2 \lesssim \frac{1}{b^2|\log b|^2} \quad \text{for} \quad 0 \leq i \leq 1. \tag{2.55}$$

We now prove this bound. A rough estimate looses the huge gain $\frac{1}{|\log b|^2}$, and we need to be more precise. From (2.51),

$$H^2(\Lambda T_3) = H(2\Sigma_3 + \Lambda\Sigma_3) + O\left(\frac{1}{1+y^2}\right). \tag{2.56}$$

We use again (2.40), together with (2.43), (2.36), to obtain:

$$H\Sigma_3 = -2HT_2 + H\Lambda T_2 + O\left(\frac{|\log y|^3}{y^2}\right) = \Lambda\Sigma_2 + O\left(\frac{|\log y|^5}{y^2}\right),$$

$$H\Lambda\Sigma_3 = 2\Lambda\Sigma_2 + \Lambda^2\Sigma_2 + O\left(\frac{|\log y|^5}{y^2}\right)$$

and injecting this into (2.56) with (2.35) yields:

$$\int_{y\leq 2B_1} |H^2(\Lambda T_3)|^2 \lesssim \int_{y\leq 2B_1}\left|\frac{y^2}{1+y^2}\left(\mathbf{1}_{y\leq 1} + \frac{1+\log(y\sqrt{b})}{|\log b|}\mathbf{1}_{1\leq y\leq 6B_0}\right) + \frac{(\log y)^2}{y^2}\mathbf{1}_{y\geq 6B_0}\right|^2 \lesssim \frac{1}{b^2|\log b|^2}$$

and (2.55) is proved.

Step 6. Estimation of the error.

We are in position to estimate the error Ψ_b. According to our construction, we have from (2.12)

$$\Psi_b = b^2\Sigma_b + b^4\Lambda T_3 + R_1(T_1, T_2, T_3) \tag{2.57}$$

We then study the last term, the others being already estimated. The bounds (2.21), (2.20), (2.39) and (2.50) yield the bound for $y \le 2B_1$, $0 \le i \le 4$, and $0 \le j \le 5$:

$$\begin{aligned}\left|\frac{d^iR_1(y)}{dy^i}\right| &\lesssim b^4\left(y^{4-i}\mathbf{1}_{y\le 1} + b^jy^{2(j+1)-i}\left(1+|\mathrm{log}y|^2\right)\mathbf{1}_{y\ge 1}\right)\\ &\lesssim b^4\left(y^{4-i}\mathbf{1}_{y\le 1} + y^{2-i}|\mathrm{log}b|^C\mathbf{1}_{y\ge 1}\right).\end{aligned}$$

Hence:

$$\begin{aligned}\int_{y\le 2B_1}|HR_1|^2 &\lesssim b^8|\mathrm{log}b|^C\int_{y\le 2B_1}1 \lesssim b^6|\mathrm{log}b|^C,\\ \int_{y\le 2B_1}\frac{1+|\mathrm{log}y|^2}{1+y^4}|HR_1|^2 &\lesssim b^8|\mathrm{log}b|^C\int_{y\le 2B_1}\frac{1+|\mathrm{log}y|^2}{1+y^4} \lesssim b^8|\mathrm{log}b|^C,\\ \int_{y\le 2B_1}|H^2R_1|^2 + \frac{|R_1|^2}{1+y^8} &\lesssim b^8|\mathrm{log}b|^C\int_{y\le 2B_1}\frac{1}{1+y^4} \lesssim b^8|\mathrm{log}b|^C.\end{aligned}$$

Injecting these bounds together with (2.29), (2.53), (2.54), (2.55) into (2.57) yield (2.8), (2.9), (2.10).
We now prove the flux computation (2.11), which will be helpful for the improved modulation equations.

$$\begin{aligned}\frac{(H\Psi_b,\Phi_M)}{(\Lambda Q,\Phi_M)} &= \frac{1}{(\Lambda Q,\Phi_M)}\left[\left(-b^2c_b\chi_{\frac{B_0}{4}}\Lambda Q,\Phi_M\right) + O\left(C(M)b^3\right)\right]\\ &= -c_bb^2 + O\left(C(M)b^3\right) = -\frac{2b^2}{|\mathrm{log}b|} + O\left(\frac{b^2}{|\mathrm{log}b|^2}\right).\end{aligned}$$

Here we recall that M being chosen large enough, we assume $|b| < b^*(M)$ so that the above claim make sense. This concludes the proof of Proposition 2.1.

2.2. **Localization of the profile.** Taking a careful look at the profiles $(T_i)_{1\le i\le 3}$, we can notice that for $y \gg B_1$, Q is negligible compared to $bT_1 + b^2T_2 + b^3T_3$. Obviously, this doesn't make sense, because we look for a solution close to Q. So, we must localize the profiles, with cut-off smooth functions. For technical reasons, we use two localizations: one at B_1, another one at B_0.

Proposition 2.3 (Localization of the profile near B_1)**.** *Let a $\mathcal{C}^1$ map $s \mapsto b(s)$ be defined on $[0, s_0]$ with a priori bound $\forall s \in [0, s_0]$,*

$$0 < b(s) < b^*(M), \quad |b_s| \le 10b^2. \tag{2.58}$$

Let the localized profile

$$\tilde{Q}_b(s,y) = Q + b\tilde{T}_1 + b^2\tilde{T}_2 + b^3\tilde{T}_3 = Q + \tilde{\alpha}$$

where

$$\tilde{T}_i = \chi_{B_1}T_i, \quad 1 \le i \le 3.$$

Then

$$\partial_s\tilde{Q}_b - \Delta\tilde{Q}_b - \frac{\lambda_s}{\lambda}\Lambda\tilde{Q}_b - \tilde{Q}_b^3 = Mod(t) + \tilde{\Psi}_b \tag{2.59}$$

with

$$Mod(t) = -\left(\frac{\lambda_s}{\lambda} + b\right)\Lambda\tilde{Q}_b + (b_s + b^2)(\tilde{T}_1 + 2b\tilde{T}_2) \tag{2.60}$$

and where $\tilde{\Psi}_b$ *satisfies the bounds on* $[0, s_0]$*:*
(i) Weighted bounds:

$$\int |H\tilde{\Psi}_b|^2 \lesssim b^4|\mathrm{log}b|^2, \tag{2.61}$$

$$\int \frac{1}{1+y^8}|\tilde{\Psi}_b|^2 \lesssim b^6, \tag{2.62}$$

$$\int |H^2\tilde{\Psi}_b|^2 \lesssim \frac{b^6}{|\mathrm{log}b|^2}. \tag{2.63}$$

(ii) Flux computation: Let Φ_M *be given by* (3.2)*, then:*

$$\frac{(H\tilde{\Psi}_b, \Phi_M)}{(\Lambda Q, \Phi_M)} = -\frac{2b^2}{|\mathrm{log}b|} + O\left(\frac{b^2}{|\mathrm{log}b|^2}\right). \tag{2.64}$$

We introduce a second localization at B_0 which will be relevant for H^2 control, see the proof of Proposition 5.3.

Proposition 2.4 (Second localization)**.** *Let a* $\mathcal{C}^1$ *map* $s \mapsto b(s)$ *be defined on* $[0, s_0]$ *with a priori bound* (2.58)*. Let the localized profile*

$$\hat{Q}_b(s,y) = Q + b\hat{T}_1 + b^2\hat{T}_2 + b^3\hat{T}_3 = Q + \hat{\alpha} \tag{2.65}$$

where

$$\hat{T}_i = \chi_{B_0}T_i, \quad 1 \le i \le 3.$$

Let the radiation:

$$\zeta_b = \tilde{\alpha} - \hat{\alpha} \tag{2.66}$$

and the error

$$\partial_s\hat{Q}_b - \Delta\hat{Q}_b - \frac{\lambda_s}{\lambda}\Lambda\hat{Q}_b - \hat{Q}_b^3 = \widehat{Mod}(t) + \hat{\Psi}_b$$

with

$$\widehat{Mod}(t) = -\left(\frac{\lambda_s}{\lambda} + b\right)\Lambda\hat{Q}_b + (b_s + b^2)(\hat{T}_1 + 2b\hat{T}_2). \tag{2.67}$$

Then there hold the bounds:

$$\left\|\partial_y^i\zeta_b\right\|_{L^\infty}^2 \lesssim b^{2+i}|\mathrm{log}b|^C, \tag{2.68}$$

$$\int |H\zeta_b|^2 \lesssim b^2|\mathrm{log}b|^2, \quad \sum_{i=0}^{2}\int \frac{|\partial_y^i\zeta_b|^2}{1+y^{2(3-i)}} \lesssim b^3|\mathrm{log}b|^C, \tag{2.69}$$

$$\int |H^2\zeta_b|^2 + \sum_{i=0}^{2}\int \frac{|\partial_y^i\zeta_b|^2}{1+y^{8-2i}} \lesssim b^4|\mathrm{log}b|^C, \tag{2.70}$$

$$Supp(H\hat{\Psi}_b) \subset [0, 2B_0] \quad and \quad \int |H\hat{\Psi}_b|^2 \lesssim b^4|\mathrm{log}b|^2. \tag{2.71}$$

The proof follows similar lines as in [19] and is displayed for the reader's convenience in Appendix C.

3. **Presentation of possible solution of Theorem 1.1**

3.1. **Uniqueness of the decomposition.** We now look for a solution u of (1.1) u, which we will decompose in the form of:

$$u = \left(\tilde{Q}_{b(t)} + \varepsilon(t)\right)_{\lambda(t)}. \tag{3.1}$$

Naturally, we must fix constrains to obtain the uniqueness of this decomposition. Moreover it's crucial that the radiation term ε doesn't perturb the modulation equation (2.30) found during the construction. We will see in the subsection devoted to the modulation equations that it's the case if we have the following inequality:

$$\int |H^2(\varepsilon(t))|^2 \lesssim \frac{b^4(t)}{|\log b(t)|^2}.$$

To control sharply the radiation term ε, we need to ensure suitable orthogonality conditions with respect to the kernel of H^2. The smooth solutions of $H^2 f = 0$ are situated in $Span(\Lambda Q, T_1)$. But neither ΛQ nor T_1 is in $L^2(\mathbb{R}^4)$. Therefore, we use an approximation of the kernel, localizing both directions, with the smooth cut-off function χ_M, where $M > 0$ is a large enough constant. More precisely, we let the direction:

$$\Phi_M = \chi_M \Lambda Q - c_M H(\chi_M \Lambda Q) \tag{3.2}$$

with

$$c_M = \frac{(\chi_M \Lambda Q, T_1)}{(H(\chi_M \Lambda Q), T_1)} = c_\chi \frac{M^2}{4}(1 + o_{M\to+\infty}(1)).$$

The second term, which is a corrective term, ensures the orthogonality between Φ_M and T_1, and the orthogonality between ΛQ and $H\Phi_M$, because H is a self-adjoint operator. Furthermore, we have by construction:

$$\int |\Phi_M|^2 \lesssim |\log M|, \tag{3.3}$$

and the scalar products

$$(\Lambda Q, \Phi_M) = (-HT_1, \Phi_M) = (\chi_M \Lambda Q, \Lambda Q) = 64 \log M(1 + o_{M\to+\infty}(1)). \tag{3.4}$$

In Appendix A-C, we argue that we have coercive estimates for the operators H and H^2 under additional orthogonality conditions. As a consequence, we fix for the radiation term ε the orthogonality conditions:

$$(\varepsilon(t), \Phi_M) = (\varepsilon(t), H\Phi_M) = 0. \tag{3.5}$$

From a standard argument based on the implicit function theorem, these constrains give us the existence and the uniqueness of the decomposition (3.1). First, we have:

$$(b, \lambda) \to (u, \Phi_M) = \left(\left(\tilde{Q}_{b(t)}\right)_{\lambda(t)}, \Phi_M\right)$$

is a $\mathcal{C}^1$ map and thus:

$$\begin{aligned}
\begin{vmatrix} (\frac{\partial}{\partial\lambda}(\tilde{Q}_b)_\lambda, \Phi_M) & (\frac{\partial}{\partial b}(\tilde{Q}_b)_\lambda, \Phi_M) \\ (\frac{\partial}{\partial\lambda}(\tilde{Q}_b)_\lambda, H\Phi_M) & (\frac{\partial}{\partial b}(\tilde{Q}_b)_\lambda, H\Phi_M) \end{vmatrix}_{\lambda=1,b=0} &= \begin{vmatrix} (-\Lambda Q, \Phi_M) & 0 \\ 0 & (T_1, H\Phi_M) \end{vmatrix} \\
&= -(\Lambda Q, \Phi_M)^2 \neq 0.
\end{aligned} \tag{3.6}$$

We used the orthogonality conditions mentioned at the moment of the conception of Φ_M and the following equality:

$$\left(\frac{\partial}{\partial\lambda}(\tilde{Q}_b)_\lambda,\ \frac{\partial}{\partial b}(\tilde{Q}_b)_\lambda\right)|_{\lambda=1,b=0} = -(\Lambda Q, T_1).$$

As long as the solution remains in a fixed small neighborhood of Q for the norm $\dot{H}^1$ what will be ensured for a suitable set of initial data, the implicit function theorem ensures the existence and the uniqueness of the decomposition (3.1).

3.2. **Partial differential equation verified by the radiation and suitable energies.** From now on, we always use the last decomposition, and also its reformulation in original variables:

$$u = \frac{1}{\lambda(t)}\left(\tilde{Q}_b + \varepsilon\right)\left(t, \frac{r}{\lambda(t)}\right) = \frac{1}{\lambda(t)}\tilde{Q}_{b(t)}\left(t, \frac{r}{\lambda(t)}\right) + w(t,r). \tag{3.7}$$

We recall the correspondence between both systems of variables

$$s(t) = \int_0^t \frac{d\tau}{\lambda^2(\tau)} \quad \text{and} \quad y = \frac{r}{\lambda(t)}.$$

We give also the rescaling formulas

$$u(t,r) = \frac{1}{\lambda}v(s,y), \qquad \partial_t u = \frac{1}{\lambda^2}\left(\partial_s v - \frac{\lambda_s}{\lambda}\Lambda v\right)_\lambda.$$

We then can inject the decomposition (3.7) with the rescaled variables in Eq. (1.1) using the one of $\tilde{Q}_b$ (2.59) and obtain the following:

$$\partial_s \varepsilon - \frac{\lambda_s}{\lambda}\Lambda\varepsilon + H\varepsilon = F - Mod = \mathcal{F}, \tag{3.8}$$

where we remind that

$$H = -\Delta - V, \qquad Mod(t) = -\left(\frac{\lambda_s}{\lambda} + b\right)\Lambda\tilde{Q}_b + (b_s + b^2)(\tilde{T}_1 + 2b\tilde{T}_2)$$

and where we denoted:

$$F = -\tilde{\Psi}_b + L(\varepsilon) + N(\varepsilon), \tag{3.9}$$

where L is a linear operator coming from the difference between H and H_{B_1}:

$$L(\varepsilon) = H\varepsilon - H_{B_1}\varepsilon = 3(\tilde{Q}_b^2 - Q^2)\varepsilon \tag{3.10}$$

with

$$H_{B_1} = -\Delta - 3\tilde{Q}_b^2$$

and the nonlinear term is given by:

$$N(\varepsilon) = 3\tilde{Q}_b\varepsilon^2 + \varepsilon^3. \tag{3.11}$$

It's important to remark that we used here the localization of the profiles near B_1. At the end of this subsection, we will introduce in the same way some new operators with the second localization near B_0. Before rewriting (3.8) with the original variables, we introduce the suitable norms for our study:

- Energy norm

$$\mathcal{E}_1 = \int |\nabla\varepsilon|^2 \tag{3.12}$$

- Higher order Sobolev norms

$$\mathcal{E}_2 = \int |H\varepsilon|^2 = \int |\varepsilon_2|^2, \qquad \mathcal{E}_4 = \int |H^2\varepsilon|^2 = \int |\varepsilon_4|^2 \tag{3.13}$$

  with $\varepsilon_i = H^i\varepsilon$ for $i \in \{2;4\}$.

To work with the original variables, we recall the notation:

$$f_\lambda(y) = \frac{1}{\lambda} f(\frac{r}{\lambda}).$$

Furthermore we must adapt this notation for the potential term namely because of its quadratic nature:

$$\tilde{V}(y) = \frac{1}{\lambda^2} V(\frac{r}{\lambda}).$$

Then (3.8) becomes:

$$\partial_t w + H_\lambda w = \frac{1}{\lambda^2}\mathcal{F}_\lambda. \tag{3.14}$$

We define $w_i = H_\lambda^i w$ for $i \in \{2;4\}$ which verify respectively:

$$\partial_t w_2 + H_\lambda w_2 = -\partial_t \tilde{V} w + H_\lambda \left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right), \tag{3.15}$$

$$\partial_t w_4 + H_\lambda w_4 = -\partial_t \tilde{V} w_2 - H_\lambda \left(\partial_t \tilde{V} w\right) + H_\lambda^2 \left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right). \tag{3.16}$$

We have also by substitution:

$$\lambda^2 \mathcal{E}_2 = \int |Hw|^2 = \int |w_2|^2, \qquad \lambda^6 \mathcal{E}_4 = \int |H^2 w|^2 = \int |w_4|^2. \tag{3.17}$$

We now focus onto the localization near B_0. Using the definition of the radiation ζ_b (2.66), we obtain the new unique decomposition:

$$u = (\hat{Q}_b + \hat{\varepsilon})_\lambda \quad \text{ie} \quad \hat{\varepsilon} = \varepsilon + \zeta_b. \tag{3.18}$$

Thus, with this localization, we have:

$$\partial_s \hat{\varepsilon} - \frac{\lambda_s}{\lambda}\Lambda\hat{\varepsilon} + \hat{H}\hat{\varepsilon} = \hat{F} - \widehat{Mod} = \hat{\mathcal{F}}, \tag{3.19}$$

where we remind that

$$\hat{H} = -\Delta - \hat{V}, \qquad \widehat{Mod}(t) = -\left(\frac{\lambda_s}{\lambda} + b\right)\Lambda\hat{Q}_b + (b_s + b^2)(\hat{T}_1 + 2b\hat{T}_2)$$

and where we have noticed

$$\hat{F} = -\hat{\Psi}_b + \hat{L}(\hat{\varepsilon}) + \hat{N}(\hat{\varepsilon}), \tag{3.20}$$

where $\hat{L}$ is a linear operator coming from the difference between $\hat{H}$ and $\hat{H}_{B_1}$:

$$\hat{L}(\varepsilon) = \hat{H}\hat{\varepsilon} - \hat{H}_{B_1}\hat{\varepsilon} = 3(\hat{Q}_b^2 - \hat{Q}^2)\hat{\varepsilon} \tag{3.21}$$

with

$$\hat{H}_{B_1} = -\Delta - 3\hat{Q}_b^2$$

and a last purely nonlinear term:

$$\hat{N}(\hat{\varepsilon}) = 3\hat{Q}_b\hat{\varepsilon}^2 + \hat{\varepsilon}^3. \tag{3.22}$$

We define likewise the operators $\hat{\varepsilon}_2$, $\hat{w}$ and $\hat{w}_2$, which come of course respectively from ε_2, w, and w_2. The energy at H^2 level becomes:

$$\hat{\mathcal{E}}_2 = \int |\hat{\varepsilon}_2|^2. \tag{3.23}$$

Moreover, with the bounds of the radiation (2.69), we can measure the difference between both energies at H^2 level.

$$\hat{\mathcal{E}_2} \lesssim \mathcal{E}_2 + \int |H\zeta_b|^2 \lesssim \mathcal{E}_2 + b^2|\text{log}b|^2. \tag{3.24}$$

Finally, $\hat{w}_2$ verifies the following partial differential equation:

$$\partial_t \hat{w}_2 + \hat{H}_\lambda \hat{w}_2 = -\partial_t \tilde{V} \hat{w} + \hat{H}_\lambda \left(\frac{1}{\lambda^2} \hat{\mathcal{F}}_\lambda \right). \tag{3.25}$$

3.3. **Modulation equations.** With the choice of orthogonality conditions (3.5), we can now measure the error made by taking $b = \frac{-\lambda_s}{\lambda}$ and $b_s = -b^2 \left(1 + \frac{2}{|\log b|}\right)$. This estimations are in the core of our proof. The proof is the same as that of [19], with the exception of a very small difference with the linear and nonlinear operators $L(\varepsilon)$ and $N(\varepsilon)$, which truly brings any difficulty, because of the same interpolation bounds. In view of the importance to this lemma in our proof of Theorem 1.1, we considered useful to give an extensive proof.

Lemma 3.1 (Modulation equations)**.** *There holds the bound on the modulation parameters:*

$$\left| \frac{\lambda_s}{\lambda} + b \right| \lesssim \frac{b^2}{|\log b|} + \frac{1}{\sqrt{\log M}} \sqrt{\mathcal{E}_4}, \tag{3.26}$$

$$\left| b_s + b^2 \left(1 + \frac{2}{|\log b|}\right) \right| \lesssim \frac{1}{\sqrt{\log M}} \left(\sqrt{\mathcal{E}_4} + \frac{b^2}{|\log b|} \right). \tag{3.27}$$

Remark 3.2. Note that this implies in the bootstrap the rough bounds:

$$|b_s| + \left| \frac{\lambda_s}{\lambda} + b \right| \le 2b^2. \tag{3.28}$$

and in particular (2.58) holds.

Proof of Lemma 3.1

Step 1. Law for b.

Let

$$V(t) = |b_s + b^2| + \left| \frac{\lambda_s}{\lambda} + b \right|.$$

We take the inner product of (3.8) with $H\Phi_M$ and estimate each terms.

$$\begin{aligned} & (\partial_s \varepsilon, H\Phi_M) + (H\varepsilon, H\Phi_M) - \left(\frac{\lambda_s}{\lambda} \Lambda \varepsilon, H\Phi_M \right) \\ = & -\left(\tilde{\Psi}_b, H\Phi_M \right) - (Mod(t), H\Phi_M) + (L(\varepsilon), H\Phi_M) + (N(\varepsilon), H\Phi_M). \end{aligned}$$

First according to our choice of orthogonality (3.5)

$$(\partial_s \varepsilon, H\Phi_M) = \partial_s (H\varepsilon, \Phi_M) = 0. \tag{3.29}$$

Then

$$(H\varepsilon, H\Phi_M) = \left(H^2 \varepsilon, \Phi_M\right) \lesssim \|H^2 \varepsilon\|_{L^2} \|\Phi_M\|_{L^2} \lesssim \sqrt{\mathcal{E}_4 logM}. \tag{3.30}$$

From the construction of the profile, (2.60) and the localization $\mathrm{Supp}(\Phi_M) \subset [0, 2M]$ from (3.2):

$$\begin{aligned} (H\left(Mod(t)\right), \Phi_M) &= -\left(b + \frac{\lambda_s}{\lambda}\right) \left(H\Lambda \tilde{Q}_b, \Phi_M\right) + \left(b_s + b^2\right) \left(H\left(\tilde{T}_1 + 2b\tilde{T}_2\right), \Phi_M\right) \\ &= -(\Lambda Q, \Phi_M)(b_s + b^2) + O\left(c(M) b |V(t)|\right). \end{aligned}$$

Using the Hardy bounds of Appendix B:

$$\left| \left(-\frac{\lambda_s}{\lambda} \Lambda \varepsilon + L(\varepsilon) + N(\varepsilon), H\Phi_M \right) \right| \lesssim C(M) b (\mathcal{E}_4 + |V(t)|).$$

We conclude from (3.4) and the fundamental flux computation (2.64):

$$\begin{aligned} b_s+b^2 &= \frac{(\tilde{\Psi}_b,H\Phi_M)}{(\Lambda Q,\Phi_M)}+O\left(\frac{\sqrt{\log M\mathcal{E}_4}}{\log M}\right)+O\left(C(M)b|V(t)|\right)\\ &= -\frac{2b^2}{|\log b|}\left(1+O\left(\frac{1}{|\log b|}\right)\right)+O\left(\sqrt{\frac{\mathcal{E}_4}{\log M}}+C(M)b|V(t)|\right)\end{aligned}$$

and (3.27) is proved.

Step 2. Degeneracy of the law for λ.

Now we take the inner product of (3.8) with Φ_M and obtain:

$$(\mathrm{Mod}(t),\Phi_M)=-(\tilde{\Psi}_b,\Phi_M)-(\partial_s\varepsilon+H\varepsilon,\Phi_M)-\left(-\frac{\lambda_s}{\lambda}\Lambda\varepsilon+L(\varepsilon)+N(\varepsilon),\Phi_M\right).$$

From our choice of orthogonality conditions (3.5):

$$(\partial_s\varepsilon+H\varepsilon,\Phi_M)=0.$$

In the same way as in the last step, using the Hardy bounds of Appendix B:

$$\left|\left(-\frac{\lambda_s}{\lambda}\Lambda\varepsilon+L(\varepsilon)+N(\varepsilon),\Phi_M\right)\right|\lesssim C(M)b\sqrt{\mathcal{E}_4}.$$

Next, we compute from (3.4) and the orthogonality (3.3):

$$\begin{aligned}(\mathrm{Mod}(t),\Phi_M) &= -\left(b+\frac{\lambda_s}{\lambda}\right)\left(\Lambda\tilde{Q}_b,\Phi_M\right)+\left(b_s+b^2\right)\left(\tilde{T}_1+2b\tilde{T}_2,\Phi_M\right)\\ &= -4\log M(1+o_{M\to+\infty}(1))\left(\frac{\lambda_s}{\lambda}+b\right)+O\left(C(M)b|V(t)|\right).\end{aligned}$$

and observe the cancellation from (2.25), (3.3):

$$\left|\left(\tilde{\Psi}_b,\Phi_M\right)\right|\lesssim b^2|(\Sigma_b,\Phi_M)|+O(C(M)b^3)=c_bb^2|(T_1,\Phi_M)|+O(C(M)b^3)=O(C(M)b^3).$$

We thus obtain the modulation equation for scaling:

$$\left|\frac{\lambda_s}{\lambda}+b\right|\lesssim b^3C(M)+bC(M)O\left(\sqrt{\mathcal{E}_4}+|V(t)|\right). \tag{3.31}$$

With (3.27), we obtain the bound

$$|V(t)|\lesssim\frac{b^2}{|\log b|}+\frac{1}{\sqrt{logM}}\sqrt{\mathcal{E}_4}.$$

Injecting this bound in (3.31) implies the refined bound (3.27). This concludes the proof of Lemma 3.1.

3.4. **Proof of Theorem 1.1.** In this section, we conclude the proof of Theorem 1.1 assuming the following a priori bounds on the solution on its maximum time interval of existence $[0,T)$, $0<T\leq+\infty$:

- Energy estimates

$$\forall t\in[0,T[\quad \mathcal{E}_1(t)\leq\delta(b^*),\qquad \mathcal{E}_2(t)\lesssim b(t)^2|\log b(t)|^5,\qquad \mathcal{E}_4(t)\lesssim\frac{b(t)^4}{|\log b(t)|^2} \tag{3.32}$$

- Link between both laws $b(t)$ and $\lambda(t)$: there exist α_1, $\alpha_2>0$ such that:

$$C(u_0)b(t)|\log b(t)|^{\alpha_1}\leq\lambda(t)\leq C'(u_0)b(t)|\log b(t)|^{\alpha_2} \tag{3.33}$$

The heart of our analysis in Section 4 will be to produce such kind of solutions. We now assume (3.32) and (3.33) and prove Theorem 1.1. The proof adapts the argument in [19] which we sketch for the convenience of the reader.

Step 1. Finite time blow-up.

Let $T \leq +\infty$ be the life time of u. From (3.32), (3.33),

$$-\frac{d}{dt}\sqrt{\lambda} = -\frac{1}{2\lambda\sqrt{\lambda}}\frac{\lambda_s}{\lambda} \gtrsim \frac{b}{\lambda\sqrt{\lambda}} \gtrsim C(u_0) > 0$$

and thus λ touches zero in finite time which implies

$$T < +\infty.$$

The bounds (3.32) and standard H^4 local well posedness theory ensure that blow-up corresponds to

$$\lambda(t) \to 0 \quad \text{as} \quad t \to T \tag{3.34}$$

and thus, with (3.33)

$$\lambda(T) = b(T) = 0. \tag{3.35}$$

Step 2. Derivation of the sharp blow-up speed.

To begin, the modulation laws become with the bound (3.32) :

$$\left| b + \frac{\lambda_s}{\lambda} \right| \lesssim \frac{b^2}{|\text{log} b|} \tag{3.36}$$

$$\left| b_s + b^2 \left(1 + \frac{2}{|\text{log} b|} \right) \right| \lesssim \frac{1}{\sqrt{\text{log} M}} \frac{b^2}{|\text{log} b|}. \tag{3.37}$$

We have defined M as a large enough constant. Let

$$B_\delta = \frac{1}{b^\delta}. \tag{3.38}$$

Since the variation of $b(s)$ is very small, we can consider the first time such that $M = B_\delta$ in (3.37). Assuming this, prove (1.8), and demonstrate afterwards that the made error is negligible for δ enough small. We have thus that (3.37) becomes :

$$\left| b_s + b^2 \left(1 + \frac{2}{|\text{log} b|} \right) \right| \lesssim \frac{1}{\sqrt{\text{log} B_\delta}} \frac{b^2}{|\text{log} b|} \lesssim \frac{b^2}{|\text{log} b|^{\frac{3}{2}}}. \tag{3.39}$$

We now integrate this in time using $\lim_{s \to +\infty} b(s) = 0$:

$$b(s) = \frac{1}{s} - \frac{2}{s \text{log} s} + O\left(\frac{1}{s|\text{log} s|^{\frac{3}{2}}} \right). \tag{3.40}$$

Using this decomposition of $b(s)$ in the modulation equation (3.36), we conclude:

$$-\frac{\lambda_s}{\lambda} = \frac{1}{s} - \frac{2}{s \text{log} s} + O\left(\frac{1}{s|\text{log} s|^{\frac{3}{2}}} \right).$$

We rewrite this as

$$\left| \frac{d}{ds} \text{log} \left(\frac{s\lambda(s)}{(\text{log} s)^2} \right) \right| \lesssim \frac{1}{s|\text{log} s|^{\frac{3}{2}}}$$

and thus integrating in time yields the existence of $\kappa(u_0) > 0$ such that:

$$\frac{s\lambda(s)}{(\log s)^2} = \frac{1}{\kappa(u_0)}\left[1 + O\left(\frac{1}{|\log s|^{\frac{3}{2}}}\right)\right].$$

Taking the log yields the bound

$$|\log\lambda| = |\log s|\left[1 + O\left(\frac{|\log\log s|}{\log s}\right)\right]$$

and thus

$$\frac{1}{s} = \kappa(u_0)\frac{\lambda}{|\log\lambda|^2}\left(1 + o(1)\right).$$

Injecting this into (3.45) yields:

$$-\lambda\lambda_t = -\frac{\lambda_s}{\lambda} = \frac{1}{s}\left(1 + o(1)\right) = \kappa(u_0)\frac{\lambda}{|\log\lambda|^2}\left(1 + o(1)\right) \tag{3.41}$$

and thus

$$-|\log\lambda|^2\lambda_t = \kappa(u_0)(1 + o(1)).$$

Integrating from t to T with $\lambda(T) = 0$ yields

$$\lambda(t) = \kappa(u_0)\frac{T - t}{|\log(T - t)|^2}\left[1 + o(1)\right],$$

and (1.8) is proved.

We now prove that the error we made is negligible. Indeed, we take the inner product of (3.8) with $H\chi_{B_\delta}\Lambda Q$ and obtain:

$$\begin{aligned} &\frac{d}{ds}\left\{(H\varepsilon, \chi_{B_\delta}\Lambda Q)\right\} - (H\varepsilon, \partial_s\chi_{B_\delta}\Lambda Q) + \frac{\lambda_s}{\lambda}(\chi_{B_\delta}\Lambda Q, H\Lambda\varepsilon) + (H^2\varepsilon, \chi_{B_\delta}\Lambda Q) \\ = &\left(H\left[-\tilde{\Psi}_b + L(\varepsilon) - N(\varepsilon) - Mod\right], \chi_{B_\delta}\Lambda Q\right). \end{aligned} \tag{3.42}$$

We must estimate all terms in this identity. First, for δ small enough, we have the rough bound:

$$|(H\varepsilon, \partial_s\chi_{B_\delta}\Lambda Q)| + |\frac{\lambda_s}{\lambda}(\chi_{B_\delta}\Lambda Q, H\Lambda\varepsilon)| + |(H[L(\varepsilon) - N(\varepsilon)], \chi_{B_\delta}\Lambda Q)| \lesssim \frac{b}{b^{C\delta}}\sqrt{\mathcal{E}_4} \lesssim \frac{b^2}{|\log b|^2}.$$

For the linear term, we have immediately:

$$|(H^2\varepsilon, \chi_{B_\delta}\Lambda Q)| \lesssim \sqrt{\mathcal{E}_4}\sqrt{|\log b|} \lesssim \frac{b^2}{\sqrt{|\log b|}}.$$

The $\tilde{\Psi}_b$ term is computed from (2.57):

$$(-H\tilde{\Psi}_b, \chi_{B_\delta}\Psi_b) = -b^2(H\Sigma_b, \chi_{B_\delta}\Psi_b) + O\left(\frac{b^3}{b^{C\delta}}\right) = b^2 c_b(\Lambda Q, \chi_{B_\delta}\Lambda Q) + O\left(\frac{b^2}{|\log b|^2}\right).$$

From (2.60), we have the following estimate for the modulation term:

$$\begin{aligned} (-HMod, \chi_{B_\delta}\Lambda Q) &= \left(\frac{\lambda_s}{\lambda} + b\right)(H\Lambda\tilde{Q}_b, \chi_{B_\delta}\Lambda Q) - (b_s + b^2)(H(\tilde{T}_1 + 2b\tilde{T}_2), \chi_{B_\delta}\Lambda Q) \\ &= (b_s + b^2)(\Lambda Q, \chi_{B_\delta}\Lambda Q) + O\left(\frac{b}{b^{C\delta}}\frac{b^2}{|\log b|}\right). \end{aligned}$$

We now inject the estimates into (3.42) and obtain:

$$(b_s + b^2)(\Lambda Q, \chi_{B_\delta}\Lambda Q) = \frac{d}{ds}\left\{(H\varepsilon, \chi_{B_\delta}\Lambda Q)\right\} - c_b b^2(\Lambda Q, \chi_{B_\delta}\Lambda Q) + O\left(\frac{b^2}{\sqrt{|\log b|}}\right)$$

which we rewrite using (2.22) and an integration by parts in time:

$$\frac{d}{ds}\left\{b-\frac{(H\varepsilon,\chi_{B_\delta}\Lambda Q)}{(\Lambda Q,\chi_{B_\delta}\Lambda Q)}\right\}+b^2\left(1+\frac{2}{|\mathrm{log} b|}\right) \tag{3.43}$$
$$= \quad O\left(\frac{b^2}{|\mathrm{log} b|^{\frac{3}{2}}}\right)+(H\varepsilon,\chi_{B_\delta}\Lambda Q)\frac{(\Lambda Q,\partial_s\chi_{B_\delta}\Lambda Q)}{(\Lambda Q,\chi_{B_\delta}\Lambda Q)^2}.$$

We now estimate:

$$\left|(H\varepsilon,\chi_{B_\delta}\Lambda Q)\frac{(\Lambda Q,\partial_s\chi_{B_\delta}\Lambda Q)}{(\Lambda Q,\chi_{B_\delta}\Lambda Q)^2}\right|\lesssim\frac{\sqrt{\mathcal{E}_4}}{b^{C\delta}}\frac{|b_s|}{b}\lesssim\frac{b^3}{b^{C\delta}},$$
$$\left|\frac{(H\varepsilon,\chi_{B_\delta}\Lambda Q)}{(\Lambda Q,\chi_{B_\delta}\Lambda Q)}\right|\lesssim\frac{\sqrt{\mathcal{E}_4}}{b^{C\delta}}\lesssim\frac{b^2}{b^{C\delta}}.$$

We inject these bounds into (3.43) and conclude that the difference between b and $\tilde b$ is given by

$$\tilde b=b-\frac{(H\varepsilon,\chi_{B_\delta}\Lambda Q)}{(\Lambda Q,\chi_{B_\delta}\Lambda Q)}=b+O\left(\frac{b^2}{|\mathrm{log} b|^2}\right) \tag{3.44}$$

and satisfies the pointwise differential control:

$$\left|\tilde b_s+\tilde b^2\left(1+\frac{2}{|\mathrm{log}\tilde b|}\right)\right|\lesssim\frac{\tilde b^2}{|\mathrm{log}\tilde b|^{\frac{3}{2}}},$$

which we rewrite :

$$\frac{\tilde b_s}{\tilde b^2\left(1+\frac{2}{|\mathrm{log}\tilde b|}\right)}+1=O\left(\frac{1}{|\mathrm{log}\tilde b|^{\frac{3}{2}}}\right).$$

We now integrate this in time using $\lim_{s\to+\infty}\tilde b(s)=0$ from (3.35), (3.44) and get:

$$\tilde b(s)=\frac{1}{s}-\frac{2}{s\mathrm{log} s}+O\left(\frac{1}{s|\mathrm{log} s|^{\frac{3}{2}}}\right)$$

and thus from (3.44):

$$b(s)=\frac{1}{s}-\frac{2}{s\mathrm{log} s}+O\left(\frac{1}{s|\mathrm{log} s|^{\frac{3}{2}}}\right). \tag{3.45}$$

This concludes the proof.

Step 3. Quantization of the focused energy.

We now turn to the proof of (1.7), (1.9) and adapt the strategy in [15]. We shall need the following bound, which is a direct consequence of our construction and (3.32):

$$\forall t\in[0,T),\quad \|\Delta\tilde u(t,x)\|_{L^2}\leq C(v_0). \tag{3.46}$$

where

$$\tilde u(t,x)=u(t,x)-\frac{1}{\lambda(t)}Q\left(\frac{x}{\lambda(t)}\right) \tag{3.47}$$

The regularity of $v(t,x)$ outside the origin is a standard consequence of parabolic regularity. Hence there exists $u^*\in\dot H^1$ such that

$$\forall R>0,\quad \nabla u(t)\to\nabla u^*\ \text{ in }\ L^2(|x|\geq R)\ \text{ as }\ t\to T.$$

Moreover, v is $\dot{H}^1$ thanks to the dissipation of the total energy, and thus recalling the decomposition (3.47) and the uniform bound (3.46):

$$\nabla \tilde{u}(t) \to \nabla u^* \text{ in } L^2 \text{ and } \Delta u^* \in L^2$$

which concludes the proof of (1.7), (1.9). This concludes the proof of Theorem 1.1.

4. Description of the initial data and bootstrap

The proof of Theorem 1.1 consists now in the demonstration of the existence of initial data, close to Q, which will satisfy the assumed bounds (3.32) and (3.33). We have already seen the condition of smallness of b to assure the uniqueness of the decomposition, through the implicit function theorem:

$$0 < b(0) < b^*(M) \ll 1. \tag{4.1}$$

In order to have some room with respect to (3.32), we fix the initial generous bounds namely:

$$|\mathcal{E}_1(0)| \leq b(0)^2, \tag{4.2}$$

and

$$|\mathcal{E}_2(0)| + |\mathcal{E}_4(0)| \leq b(0)^{10}. \tag{4.3}$$

Moreover, there is a crucial difference compared to [19]. The linear operator H possesses a negative direction ψ, source of instability, which can be harmful to the blow-up dynamics. We manage this as in [5]. We note

$$\kappa(t) = (\varepsilon(t), \psi)\,, \tag{4.4}$$

and

$$a^+ = \kappa(0) = (\varepsilon(0), \psi)\,. \tag{4.5}$$

We impose that:

$$\left|a^+\right| \leq \frac{2b(0)^{\frac{5}{2}}}{|\text{log}b(0)|}. \tag{4.6}$$

The propagation of regularity by the parabolic heat flow ensures that these estimates hold on some time interval $[0, t_1)$ together with the regularity $(\lambda, b) \in \mathcal{C}^1([0, t_1), \mathbb{R}_+^* \times \mathbb{R})$. Given a large enough universal constant $K > 0$ -independent of M-, we assume on $[0, t_1)$:

- Control of b(t):

$$0 < b(t) < 10b(0). \tag{4.7}$$

- Control of the radiation:

$$\int |\nabla \varepsilon(t)|^2 \leq 10\sqrt{b(0)}, \tag{4.8}$$

$$|\mathcal{E}_2(t)| \leq Kb^2(t)|\text{log}b(t)|^5, \tag{4.9}$$

$$|\mathcal{E}_4(t)| \leq K\frac{b^4(t)}{|\text{log}b(t)|^2}. \tag{4.10}$$

- *A priori* bound on the unstable mode

$$|\kappa(t)| \leq 2\frac{b^{\frac{5}{2}}}{|\text{log}b|}. \tag{4.11}$$

We may describe the bootstrap regime as follows:

Definition 4.1 (Exit time)**.** *Given* $a^+ \in \left[-2\frac{b(0)^{\frac{5}{2}}}{|\log b(0)|}; 2\frac{b(0)^{\frac{5}{2}}}{|\log b(0)|}\right]$*, we let* $T(a^+)$ *be the life time of the solution to* (1.1) *with initial data* (4.1)*,* (4.2)*,* (4.3) *and* (4.6)*, and let* $T_1(a^+) > 0$ *be the supremum of* $T \in (0, T(a^+))$*, such that for all* $t \in [0; T]$*, the estimates* (4.7)*,* (4.8)*,* (4.9)*,* (4.10) *and* (4.11) *hold.*

The existence of blow-up solutions in the regime described by Theorem 1.1 now follows from the following:

Proposition 4.2. *There exists* $a^+ \in \left]-2\frac{b(0)^{\frac{5}{2}}}{|\log b(0)|}; 2\frac{b(0)^{\frac{5}{2}}}{|\log b(0)|}\right[$ *such that*

$$T_1(a^+) = T(a^+) \tag{4.12}$$

and then corresponding solution of (1.1) *blows up in finite time in the regime described by Theorem 1.1*

We shall use the same strategy as in [18], [16], [5], and [19]. We will process in three times:

- First, we shall derive of suitable Lyapunov functionals at Sobolev respectively $\dot{H}^4$ and $\dot{H}^2$ levels. That is the most difficult part of the proof, particularly because of the estimates of nonlinear terms, for whose we must make a sharp study. Moreover, we shall see that it was crucial that $|a^+| \lesssim \frac{b(0)^{\frac{5}{2}}}{|\log b(0)|}$.
- Secondly, we will reintegrate this functionals to obtain improved bounds for $\mathcal{E}_2$ and $\mathcal{E}_4$. The bounds (4.7) is a direct consequence of the dissipation of energy. Thus, only the last bound (4.11) for the unstable direction can be the cause of an exit time less than the life time of the solution.
- Finally, we will study the dynamics of the unstable mode, and see that we can choose an a^+ to obtain Proposition 4.2. To ensure the existence of this solution, it is important that $|a^+| \gtrsim g(b(0))$ with $\frac{b(x)^3}{|\log b(x)|} = o\left(g(x)\right)$ as $x \to 0$, hence the choice for the bounds of a^+ is in (4.6).

5. Lyapunov monotonicities

5.1. At $\dot{H}^4$ level.

Proposition 5.1 (Lyapunov monotonicity $\dot{H}^4$)**.** *There holds:*

$$\frac{d}{dt}\left\{\frac{1}{\lambda^6}\left[\mathcal{E}_4 + O\left(\sqrt{b}\frac{b^4}{|\log b|^2}\right)\right]\right\} \leq C\frac{b}{\lambda^8}\left[\frac{\mathcal{E}_4}{\sqrt{\log M}} + \frac{b^4}{|\log b|^2} + \frac{b^2}{|\log b|}\sqrt{\mathcal{E}_4}\right] \tag{5.1}$$

for some universal constant $C > 0$ *independent of* M *and of the bootstrap constant* K *in* (4.7)*,* (4.8)*,* (4.9)*,* (4.10)*, provided* $b^*(M)$ *in* (4.1) *has been chosen small enough.*

Proof.

We recall the partial differential equations satisfied by w_2 and w_4:

$$\partial_t w_2 + H_\lambda w_2 = -\partial_t \tilde{V} w + H_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right). \tag{5.2}$$

$$\partial_t w_4 + H_\lambda w_4 = -\partial_t \tilde{V} w_2 - H_\lambda\left(\partial_t \tilde{V} w\right) + H_\lambda^2\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right). \tag{5.3}$$

Moreover, we recall the action of time derivates on rescaling:

$$\partial_t v_\lambda(r) = \frac{1}{\lambda^2}\left(\partial_s v - \frac{\lambda_s}{\lambda}\Lambda v\right)_\lambda.$$

Step 1. Energy identity.

Lemma 5.2 (Energy identity $\dot{H}^4$)**.**

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left\{w_4^2 - 2\partial_t\tilde{V}ww_4\right\}\\
=\ & -\int w_4H_\lambda w_4 + \int\left(\partial_t\tilde{V}\right)^2 w_2w - \int\partial_{tt}\tilde{V}ww_4 + \int H_\lambda\left(\partial_t\tilde{V}w\right)\partial_t\tilde{V}w\\
+\ & \int w_4H_\lambda^2\frac{1}{\lambda^2}\mathcal{F}_\lambda - \int H_\lambda\left(\partial_t\tilde{V}w\right)H_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right) - \int\partial_t\tilde{V}w_4\frac{1}{\lambda^2}\mathcal{F}_\lambda.
\end{aligned}$$

Proof. We propose here a simplification with respect to the algebra in [19]. Dissipation also allows us to sign some terms and avoid the study of suitable quadratic forms as in [5]. We compute the energy identity:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int w_4^2 &= \int w_4\partial_t w_4\\
&= \int w_4\left(-H_\lambda w_4 - \partial_t\tilde{V}w_2 - H_\lambda\left(\partial_t\tilde{V}w\right) + H_\lambda^2\frac{1}{\lambda^2}\mathcal{F}_\lambda\right).
\end{aligned}$$

We now treat separately the second and the third terms.

$$\begin{aligned}
&-\int w_4\partial_t\tilde{V}w_2\\
=\ & \int\partial_t\tilde{V}w_2\left(\partial_t w_2 + \partial_t\tilde{V}w - H_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\right)\\
=\ & \int\left(\partial_t\tilde{V}\right)^2 w_2w - \int\partial_t\tilde{V}w_2H_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right) + \int\partial_t\tilde{V}\partial_t\left(\frac{w_2^2}{2}\right)\\
=\ & \int\left(\partial_t\tilde{V}\right)^2 w_2w - \int\partial_t\tilde{V}w_2H_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right) - \frac{1}{2}\int\partial_{tt}\tilde{V}w_2^2 + \frac{1}{2}\frac{d}{dt}\left(\int\partial_t\tilde{V}w_2^2\right).
\end{aligned}$$

Now

$$\begin{aligned}
-\int w_4H_\lambda\left(\partial_t\tilde{V}w\right) &= \int H_\lambda\left(\partial_t\tilde{V}w\right)\left(\partial_t w_2 + \partial_t\tilde{V}w - H_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\right)\\
&= \int H_\lambda\left(\partial_t\tilde{V}w\right)\left(\partial_t\tilde{V}w - H_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\right)\\
&+ \frac{d}{dt}\int H_\lambda\left(\partial_t\tilde{V}w\right)w_2 - \int\partial_t\left[H_\lambda\left(\partial_t\tilde{V}w\right)\right]w_2.
\end{aligned}$$

The last term becomes

$$-\int\partial_tH_\lambda\left(\partial_t\tilde{V}w\right)w_2 = \int\left(\partial_t\tilde{V}\right)^2 w_2w - \int\partial_{tt}\tilde{V}ww_4 - \int\partial_t\tilde{V}\partial_t ww_4,$$

and

$$
\begin{aligned}
-\int \partial_t\tilde{V}\partial_t w w_4 &= \int \partial_t\tilde{V}\left(w_2-\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)w_4\\
&= -\int \partial_t\tilde{V}\frac{1}{\lambda^2}\mathcal{F}_\lambda w_4+\int \partial_t\tilde{V}w_2\left(-\partial_t w_2-\partial_t\tilde{V}w+H_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\right)\\
&= -\int \partial_t\tilde{V}\frac{1}{\lambda^2}\mathcal{F}_\lambda w_4+\int \partial_t\tilde{V}w_2\left(-\partial_t\tilde{V}w+H_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\right)\\
&- \frac{1}{2}\frac{d}{dt}\left(\int \partial_t\tilde{V}w_2^2\right)+\frac{1}{2}\int \partial_{tt}\tilde{V}w_2^2.
\end{aligned}
$$

In the following steps, we estimate each term of Lemma 5.2 in order to prove Proposition 5.1.

Step 2. Lower order quadratic terms.
We have from (2.16) and (5.4), and the modulation equations, the bounds:

$$
|\partial_t\tilde{V}|\lesssim \frac{b}{\lambda^4}\frac{1}{1+y^4}\qquad |\partial_{tt}\tilde{V}|\lesssim \frac{b}{\lambda^6}\frac{1}{1+y^4}. \tag{5.4}
$$

Using (A.3), we obtain

$$
\begin{aligned}
-\int H_\lambda w_4 w_4 = -\frac{1}{\lambda^8}\int H\varepsilon_4\varepsilon_4 &\lesssim \frac{1}{\lambda^8}\left(\int \varepsilon_4\psi\right)^2\lesssim \frac{\zeta^2}{\lambda^8}\left(\int \varepsilon\psi\right)^2\\
&\lesssim \frac{1}{\lambda^8}\kappa^2\lesssim \frac{b}{\lambda^8}\frac{b^4}{|\log b|^2}.
\end{aligned}
$$

Remark that:

$$
\int H_\lambda\left(\partial_t\tilde{V}w\right)\partial_t\tilde{V}w=\int\left(\partial_t\tilde{V}\right)^2 w_2 w-\int \Delta\left(\partial_t\tilde{V}\right)\partial_t\tilde{V}w^2-2\int \partial_{rt}\tilde{V}\partial_t\tilde{V}\partial_r w w.
$$

We treat now the two following terms:

$$
\begin{aligned}
&\left|2\int\left(\partial_t\tilde{V}\right)^2 w_2 w-\int \partial_{tt}\tilde{V}ww_4-\int \Delta\left(\partial_t\tilde{V}\right)\partial_t\tilde{V}w^2-2\int \partial_{rt}\tilde{V}\partial_t\tilde{V}\partial_r w w\right|\\
\lesssim\ & \frac{b^2}{\lambda^8}\int\left(\frac{\varepsilon\varepsilon_2}{1+y^8}+\frac{\varepsilon\varepsilon_4}{1+y^4}+\frac{\varepsilon^2}{1+y^{10}}+\frac{\varepsilon\partial_y\varepsilon}{1+y^9}\right)\lesssim \frac{b^2}{\lambda^8}b^4|\log b|^C.
\end{aligned}
$$

The last inequality comes from Cauchy-Schwarz and the bounds (B.1) and (B.3). Finally, we estimate the boundary term in time

$$
\left|\int \partial_t\tilde{V}ww_4\right|\lesssim \frac{b}{\lambda^6}\left(\int\frac{\varepsilon^2}{1+y^8}\right)^{\frac{1}{2}}\left(\int \varepsilon_4^2\right)^{\frac{1}{2}}\lesssim \frac{b}{\lambda^6}b^4|\log b|^C\lesssim \frac{\sqrt{b}}{\lambda^6}\frac{b^4}{|\log b|^2}.
$$

Step 3. Further use of dissipation.
First, we claim the following bounds:

$$
\int \frac{1}{1+y^8}\mathcal{F}^2 \lesssim \left[\frac{b^4}{|\log b|^2}+\frac{\mathcal{E}_4}{\log M}\right] \tag{5.5}
$$

$$
\int |H^2\mathcal{F}|^2 \lesssim b^2\left[\frac{b^4}{|\log b|^2}+\frac{\mathcal{E}_4}{\log M}\right]. \tag{5.6}
$$

Thus,

$$\begin{aligned}
&\left|\int w_4 H_\lambda^2 \frac{1}{\lambda^2}\mathcal{F}_\lambda - \int H_\lambda\left(\partial_t\tilde{V}w\right)H_\lambda\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right) - \int \partial_t\tilde{V}w_4\frac{1}{\lambda^2}\mathcal{F}_\lambda\right| \\
\lesssim\ & \frac{1}{\lambda^8}\left\{\int\left(\varepsilon_4^2 + b^2\frac{\varepsilon^2}{1+y^8}\right)^{\frac{1}{2}}\left(\int |H^2\mathcal{F}|^2\right)^{\frac{1}{2}} + b\left(\int\varepsilon_4^2\right)^{\frac{1}{2}}\left(\int\frac{1}{1+y^8}\mathcal{F}^2\right)^{\frac{1}{2}}\right\} \\
\lesssim\ & \frac{b}{\lambda^8}\left[\frac{\mathcal{E}_4}{\sqrt{\log M}} + \frac{b^4}{|\log b|^2} + \frac{b^2}{|\log b|}\sqrt{\mathcal{E}_4}\right].
\end{aligned}$$

This concludes the proof of the Proposition 5.1. We now turn of the proof of (5.5) and (5.6). We recall that

$$\mathcal{F} = -\tilde{\Psi}_b - Mod(t) + L(\varepsilon) + N(\varepsilon).$$

Step 4. $\tilde{\Psi}_b$ terms.
The contribution of the $\tilde{\Psi}_b$ terms in (5.5) and (5.6) has already been proved in Lemma 2.3. For that matter, the construction of the approximated solution by the profiles $(T_i)_{1\leq i\leq 3}$ has been made to obtain these good estimates.

Step 5. $Mod(t)$ terms.
Recall the definition (2.60) of $Mod(t)$.

$$Mod(t) = -\left(\frac{\lambda_s}{\lambda} + b\right)\Lambda\tilde{Q}_b + (b_s + b^2)(\tilde{T}_1 + 2b\tilde{T}_2).$$

With the modulation equation (3.26) and (3.27), we have:

$$\left|\frac{\lambda_s}{\lambda} + b\right|^2 + \left|b_s + b^2\right|^2 \lesssim \frac{b^4}{|\log b|^2} + \frac{\mathcal{E}_4}{\log M}.$$

But

$$\int\frac{1}{1+y^8}|\Lambda\tilde{Q}_b|^2 + \int\frac{1}{1+y^8}|\tilde{T}_1 + 2b\tilde{T}_2|^2 \lesssim 1.$$

Now,

$$\int |H^2\Lambda\tilde{Q}_b|^2 + \int |H^2(\tilde{T}_1 + 2b\tilde{T}_2)|^2 \lesssim b^2. \tag{5.7}$$

Indeed :

$$\int\left|H^2\tilde{T}_1\right|^2 \lesssim \int_{B_1\leq y\leq 2B_1}\left|\frac{\log y}{y^4}\right| \lesssim \frac{|\log b|^2}{B_1^4} \lesssim b^2 \tag{5.8}$$

There is a whole proof of the estimate for $\tilde{T}_2$ in [19]. Here is a summary of this demonstration. With the definition of $\tilde{T}_2$, and the bound (2.38) of T_2, we have :

$$\int |H^2\tilde{T}_2|^2 \lesssim \left[\int_{B_1\leq y\leq 2B_1}\left|\frac{y^2}{y^4}\right|^2 + \int_{y\leq 2B_1}|H\Sigma_2|^2\right] \lesssim 1 + \int_{y\leq 2B_1}|H\Sigma_2|^2. \tag{5.9}$$

From the construction of the radiation Σ_b and the definition of Σ_2, we compute:

$$\begin{aligned}
H\Sigma_2 &= H\Sigma_b + H(T_1 - \Lambda T_1) + O\left(\frac{y|\log y|^2}{1+y^5}\right) \\
&= \frac{1}{|\log b|}O\left(\frac{1}{1+y^2}\mathbf{1}_{y\leq 3B_0}\right) + H(T_1 - \Lambda T_1) + O\left(\frac{y|\log y|^2}{1+y^5}\right).
\end{aligned}$$

But

$$\begin{aligned} HT_1 - H\Lambda T_1 &= HT_1 - \left(2HT_1 + \Lambda HT_1 - \frac{\Lambda V}{y^2}T_1\right) \\ &= O\left(\frac{\log y}{1+y^3}\right). \end{aligned}$$

We thus conclude:

$$\int |H\Sigma_2|^2 \lesssim \frac{1}{|\log b|^2}\int_{y\le 2B_0}\frac{1}{1+y^4} + \int_{y\le 2B_1}\frac{|\log y|^4}{1+y^8} \lesssim 1,$$

and the contribution of $Mod(t)$ terms to (5.5) and (5.6) are small enough.

Step 6. Small linear term $L(\varepsilon)$.
We recall the expression of $L(\varepsilon)$:

$$L(\varepsilon) = 3\left(\tilde{Q}_b^2 - Q^2\right)\varepsilon.$$

We have, with the rough bounds (2.20), (2.39) and (2.50):

$$\left|3\left(\tilde{Q}_b^2 - Q^2\right)\right| \lesssim b$$

and, with (B.3)

$$\int \frac{1}{1+y^8}|L(\varepsilon)|^2 \lesssim b^2\int\frac{1}{1+y^8}\varepsilon^2 \lesssim \frac{b^4}{|\log b|^2}.$$

Let us study the second estimate. In order to do that, let

$$g = 3\left(\tilde{Q}_b^2 - Q^2\right). \tag{5.10}$$

We have the following bound:

$$|H^j\left(\partial_y^i g\right)| \lesssim b\frac{y^k(1+|\log y|)}{1+y^{2+i+2j+k}},\ 0\le i\le 4,\ 0\le j\le 1 \tag{5.11}$$

where

$$k = \max\{0; 2-i-2j\}.$$

So, with the bounds (5.11) and those of Lemma B.1, we obtain:

$$\begin{aligned} &\int |H^2(L(\varepsilon))|^2 = \int |H^2(g\varepsilon)|^2 = \int |H\left(gH\varepsilon - \varepsilon\Delta g - \partial_y g\partial_y\varepsilon\right)|^2 \\ \lesssim\ & \int |gH^2(\varepsilon)|^2 + \int |\Delta g H\varepsilon|^2 + \int |\partial_y g\partial_y(H\varepsilon)|^2 \\ +\ & \int |\Delta g H\varepsilon|^2 + \int |\varepsilon\Delta^2 g|^2 + \int |\partial_y(\Delta g)\partial_y\varepsilon|^2 \\ +\ & \int |H(\partial_y g)\partial_y\varepsilon|^2 + \int |\partial_y g\Delta(\partial_y\varepsilon)|^2 + \int |\partial_{yy}g\partial_{yy}\varepsilon|^2 \\ \lesssim\ & \frac{b^6}{|\log b|^2}. \end{aligned}$$

The proof is similar to that of the forthcoming estimates of the nonlinear terms and therefore left to the reader.

Step 7 Nonlinear term $N(\varepsilon)$.

We recall the expression of $N(\varepsilon)$:

$$N(\varepsilon) = 3\tilde{Q}_b\varepsilon^2 + \varepsilon^3.$$

We have, with the rough bounds (2.20), (2.39) and (2.50):

$$\left|3\left(\tilde{Q}_b(y) - Q(y)\right)\right| \lesssim |b|(1 + |\log y|)$$

and, with (B.6)

$$\begin{aligned} &\int \frac{1}{1+y^8}\left|3(\tilde{Q}_b - Q + Q)\varepsilon^2\right|^2 \lesssim b^2\int\frac{1+|\log y|^2}{1+y^8}\varepsilon^4 + \int\frac{\varepsilon^4}{1+y^{12}} \\ \lesssim\ & b^2\|\varepsilon\|_{L^\infty}^4\int\frac{1+|\log y|^2}{1+y^8} + \left\|\frac{\varepsilon}{1+y^2}\right\|_{L^\infty}^2\int\frac{\varepsilon^2}{1+y^8} \lesssim \frac{b^4}{|\log b|^2}\end{aligned}$$

and to conclude:

$$\int\frac{1}{1+y^8}\left|\varepsilon^3\right|^2 \lesssim \|\varepsilon\|_{L^\infty}^6\int\frac{1}{1+y^8} \lesssim \frac{b^4}{|\log b|^2}.$$

For the second bound, let us compute $H(\varepsilon^3)$

$$\begin{aligned} H(\varepsilon^3) &= \varepsilon^2 H(\varepsilon) - 2\partial_y(\varepsilon^2)\partial_y\varepsilon - \Delta(\varepsilon^2)\varepsilon \\ &= \varepsilon^2 H(\varepsilon) - 2(\partial_y\varepsilon)^2\varepsilon - \left(\varepsilon\partial_{yy}\varepsilon + (\partial_y\varepsilon)^2 + 3\frac{\varepsilon\partial_y\varepsilon}{y}\right)\varepsilon \\ &= \varepsilon^2 H(\varepsilon) - 3(\partial_y\varepsilon)^2\varepsilon - \varepsilon^2\Delta\varepsilon \\ &= 2\varepsilon^2 H(\varepsilon) + V\varepsilon^3 - 3\varepsilon(\partial_y\varepsilon)^2.\end{aligned}$$

Now, we treat each term separately. First:

$$\begin{aligned} &\int|H(\varepsilon^2 H(\varepsilon))|^2 \lesssim \int\varepsilon^4|H^2(\varepsilon)|^2 + \int\Delta(\varepsilon^2)^2 H(\varepsilon)^2 + \int|\partial_y(\varepsilon^2)|^2|\partial_y H(\varepsilon)|^2 \\ \lesssim\ & \|\varepsilon\|_{L^\infty}^4\mathcal{E}_4 + \left(\|\varepsilon\|_{L^\infty_{y\geq 1}}^2\|\partial_{yy}\varepsilon\|_{L^\infty_{y\geq 1}}^2 + \|\partial_y\varepsilon\|_{L^\infty}^4 + \|\partial_y\varepsilon\|_{L^\infty_{y\geq 1}}^2\left\|\frac{\varepsilon}{y}\right\|_{L^\infty_{y\geq 1}}^2\right)\mathcal{E}_2 \\ +\ & \left(\|\varepsilon\|_{L^\infty_{y\leq 1}}^2\|y\partial_{yy}\varepsilon\|_{L^\infty_{y\leq 1}}^2 + \|\partial_y\varepsilon\|_{L^\infty_{y\leq 1}}^2\|\varepsilon\|_{L^\infty_{y\leq 1}}^2 + \left\|\varepsilon(1+|\log y|^2)\right\|_{L^\infty_{y\geq 1}}^2\|y\partial_y\varepsilon\|_{L^\infty_{y\geq 1}}^2\right)\mathcal{E}_4 \\ \lesssim\ & \frac{b^6}{|\log b|^2}.\end{aligned}$$

Secondly, using that

$$|\partial_y^i V| \lesssim \frac{1}{1+y^{4+i}} \qquad 0 \leq i \leq 2, \tag{5.12}$$

we have:

$$\begin{aligned} &\int|H(V\varepsilon^3)|^2 \lesssim \int|H(V)|^2\varepsilon^6 + \int\Delta(\varepsilon^3)^2V^2 + \int|\partial_y(\varepsilon^3)|^2|\partial_y V|^2 \\ \lesssim\ & \|\varepsilon\|_{L^\infty}^2\left(\left\|\frac{\varepsilon}{1+y}\right\|_{L^\infty}^2\left\|\frac{\varepsilon}{1+y^2}\right\|_{L^\infty}^2 + \|\partial_y\varepsilon\|_{L^\infty}^4 + \left\|\frac{\varepsilon}{1+y^2}\right\|_{L^\infty}^2\|\partial_y\varepsilon\|_{L^\infty}^2\right)\int\frac{1}{1+y^6} \\ +\ & \|\varepsilon\|_{L^\infty}^4\int\frac{|\partial_{yy}\varepsilon|^2}{1+y^8} \lesssim \frac{b^6}{|\log b|^2}.\end{aligned}$$

Lastly

$$\int |H(\varepsilon(\partial_y\varepsilon)^2)|^2 \lesssim \int |H(\varepsilon)|^2|(\partial_y\varepsilon)^2)|^2 + \int \varepsilon^2 \left|\Delta((\partial_y\varepsilon)^2)\right|^2 + \int \partial_y\varepsilon^2 \left|\partial_y((\partial_y\varepsilon)^2)\right|^2$$

$$\lesssim \|\partial_y\varepsilon\|^4_{L^\infty} \mathcal{E}_2 + \int \varepsilon^2 \left|\Delta((\partial_y\varepsilon)^2)\right|^2 \lesssim \frac{b^6}{|\log b|^2} + \int \varepsilon^2 \left|\Delta((\partial_y\varepsilon)^2)\right|^2.$$

But

$$\begin{aligned}
& \int \varepsilon^2 \left|\Delta((\partial_y\varepsilon)^2)\right|^2 \\
= & \int \varepsilon^2 \left(|\partial_{yy}\varepsilon|^2 + \partial_y\varepsilon\partial_y^3\varepsilon + 3\frac{\partial_y\varepsilon\partial_{yy}\varepsilon}{y}\right)^2 \\
= & \int \varepsilon^2 \left(\partial_{yy}\varepsilon\left(H(\varepsilon) + V\varepsilon\right) + \partial_y\varepsilon\partial_y^3\varepsilon\right)^2 \\
\lesssim & \|\varepsilon\|^2_{L^\infty}\left(\|y\partial_{yy}\varepsilon\|^2_{L^\infty_{y\le 1}} \mathcal{E}_4 + \|\partial_{yy}\varepsilon\|^2_{L^\infty_{y\ge 1}} \mathcal{E}_2 + \|\varepsilon\|^2_{L^\infty} \int \frac{|\partial_{yy}\varepsilon|^2}{1+y^8}\right) \\
+ & \left\|\varepsilon(1+|\log y|^2)\right\|^2_{L^\infty_{y\ge 1}} \|y\partial_y\varepsilon\|^2_{L^\infty_{y\ge 1}} \mathcal{E}_4 + \|\varepsilon\|^2_{L^\infty_{y\le 1}} \|\partial_y\varepsilon\|^2_{L^\infty_{y\le 1}} \int_{y\le 1} |\partial_y^3\varepsilon|^2 \\
\lesssim & \frac{b^6}{|\log b|^2}.
\end{aligned}$$

Thus,

$$\int |H^2(\varepsilon^3)|^2 \lesssim \frac{b^6}{|\log y|^2}. \tag{5.13}$$

Treat now the other contribution of $N(\varepsilon)$ in the bound (5.6). Let

$$f = 3\left(\tilde{Q}_b - Q\right).$$

We have the following bounds:

$$|\partial_y^i f| \lesssim b\frac{y^{2-i}(1+|\log y|)}{1+y^2} + \frac{1}{1+y^{2+i}}, \quad 0 \le i \le 2 \tag{5.14}$$

$$|H(\partial_y^j f)| \lesssim b\frac{y^k}{1+y^{2+j+k}} + \frac{1}{1+y^{4+j}}, \quad 0 \le j \le 2 \tag{5.15}$$

where

$$k = \begin{cases} 1 & \text{for } \; j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let us compute $H(f\varepsilon^2)$

$$\begin{aligned}
H(f\varepsilon^2) &= H(\varepsilon^2)f - \varepsilon^2\Delta f - 2\partial_y f\partial_y(\varepsilon^2) \\
&= H(\varepsilon^2)f + \varepsilon^2 Hf - 2\partial_y f\partial_y(\varepsilon^2) + Vf\varepsilon^2
\end{aligned} \tag{5.16}$$

In the same way as in the last proof, we treat each term separately. First:

$$H\left(H(\varepsilon^2)f\right) = H^2(\varepsilon^2)f - \Delta f H(\varepsilon^2) - \partial_y f\partial_y H(\varepsilon^2). \tag{5.17}$$

Let us estimate the three components using that:

$$H(\varepsilon^2) = 3\varepsilon H(\varepsilon) + 2V\varepsilon^2 - 2\left(\partial_y\varepsilon\right)^2 \tag{5.18}$$

and,

$$\begin{aligned} H^2(\varepsilon^2) &= 3\left(\varepsilon H^2(\varepsilon)+|H(\varepsilon)|^2+V\varepsilon H(\varepsilon)-2\partial_y\varepsilon\partial_y H(\varepsilon)\right) \\ &+ 2\left(HV\varepsilon^2-V\Delta(\varepsilon^2)-2\partial_y V\partial_y(\varepsilon^2)\right) \\ &- 2\left(3\partial_y\varepsilon H(\partial_y\varepsilon)+2V(\partial_y\varepsilon)^2-2\left(\partial_{yy}\varepsilon\right)^2\right) \end{aligned} \tag{5.19}$$

and moreover,

$$H(\partial_y\varepsilon)=\partial_y H(\varepsilon)-\frac{\partial_y\varepsilon}{y^2}-\partial_y V\varepsilon \tag{5.20}$$

(5.17), (5.18), (5.19) and (5.20) together with the bounds (5.14), (5.15), (5.12) and those of Lemma B.1 imply:

$$\int |H\left(H(\varepsilon^2)f\right)|^2 \lesssim \frac{b^6}{|\log b|^2} \tag{5.21}$$

Let us study the second term of (5.16) .

$$H\left(\varepsilon^2 Hf\right)=\varepsilon^2H^2(f)-Hf\Delta(\varepsilon^2)-2\partial_y(Hf)\partial_y(\varepsilon^2). \tag{5.22}$$

The bounds (5.14), (5.15) and Lemma B.1 yield

$$\int |H\left(\varepsilon^2 Hf\right)|^2 \lesssim \frac{b^6}{|\log b|^2}. \tag{5.23}$$

We estimate the two last terms in (5.16) in the same way. This concludes the proof of (5.6) and thus of Proposition 5.1.

5.2. **At $\dot{H}^2$ level.** For the H^2 level, we use the profile $\hat{Q}_b$ localized near B_0. The description of this localization and the estimates of the new error generated by these are given by Lemma 2.4 and Section 3.2. We recall the equation verified by $\hat{w}_2$:

$$\partial_t\hat{w}_2+\hat{H}_\lambda\hat{w}_2=-\partial_t\tilde{V}\hat{w}+\hat{H}_\lambda\left(\frac{1}{\lambda^2}\hat{\mathcal{F}}_\lambda\right). \tag{5.24}$$

Proposition 5.3 (Lyapunov monotonicity $\dot{H}^2$)**.** *There hold:*

$$\frac{d}{dt}\frac{\hat{\mathcal{E}}_2}{\lambda^2}\lesssim\frac{b^3|\log b|^2}{\lambda^4} \tag{5.25}$$

Proof.

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\frac{\hat{\mathcal{E}}_2}{\lambda^2} &= \int \hat{w}_2\left[-H_\lambda\hat{w}_2-\partial_t\tilde{V}\hat{w}+H_\lambda\left(\frac{1}{\lambda^2}\hat{\mathcal{F}}_\lambda\right)\right] \\ &= -\int \hat{w}_2H_\lambda\hat{w}_2-\int\partial_t\tilde{V}\hat{w}\hat{w}_2+\int\hat{w}_2H_\lambda\left(\frac{1}{\lambda^2}\hat{\mathcal{F}}_\lambda\right). \end{aligned} \tag{5.26}$$

We study each term separately:

To begin, in agreement with (A.3), the *a priori* bound (4.11) on the unstable direction, and the bounds (2.70) we have

$$\begin{aligned} -\int\hat{w}_2H_\lambda\hat{w}_2 &\lesssim \frac{1}{\lambda^4}(\hat{\varepsilon}_2,\psi)^2 \\ &\lesssim \frac{1}{\lambda^4}\left[(\varepsilon_2,\psi)^2+(H\zeta_b,\psi)^2\right] \\ &\lesssim \frac{1}{\lambda^4}\left[\kappa^2+\|H^2\zeta_b\|_{L^2}^2\right]\lesssim\frac{b^3|\log b|^2}{\lambda^4}. \end{aligned} \tag{5.27}$$

The second term is a lower order quadratic term. Recall that:

$$|\partial_t\tilde{V}|\lesssim\frac{b}{\lambda^4}\frac{1}{1+y^4}.$$

Hence, using Cauchy-Schwarz, with the *a priori* bound (4.9), (3.24) measuring the difference between the two energies at $\dot{H}^2$ level, and the bounds (2.70) and (B.1)

$$\left|\int\partial_t\tilde{V}\hat{w}\hat{w}_2\right|\lesssim\frac{b}{\lambda^4}\hat{\mathcal{E}}_2^{\frac12}\left(\int\frac{|\varepsilon|^2+|\zeta_b|^2}{1+y^8}\right)^{\frac12}\lesssim\frac{b}{\lambda^4}b^3|\log b|^C. \tag{5.28}$$

For the last, we don't use exactly the same strategy as for the control $\dot{H}^4$. Indeed, for the term of error, the term of modulation, the global L^2 bounds for $\hat{\varepsilon}_2$ we have are too rough. We must then improve them, as both terms are localized for $y\leq 2B_0$. So:

$$\begin{aligned}\int_{y\leq 2B_0}|\hat{\varepsilon}_2|^2&\lesssim B_0^4|\log b|^2\int\frac{|\varepsilon|^2}{(1+y^4)|\log y|^2}+\int|H\zeta_b|^2\\&\lesssim C(M)b^2+b^2|\log b|^2\lesssim b^2|\log b|^2.\end{aligned} \tag{5.29}$$

The term of error $\hat{\Psi}_b$ is now estimated using (2.71) and the improved bound (5.29):

$$|(\hat{\varepsilon}_2,H\hat{\Psi}_b)|\lesssim\|H\hat{\Psi}_b\|_{L^2}\|\hat{\varepsilon}_2\|_{L^2(y\leq 2B_0)}\lesssim\left(b^4|\log b|^2b^2|\log b|^2\right)^{\frac12}\lesssim b^3|\log b|^2. \tag{5.30}$$

We next estimate from (2.20), (2.38):

$$\int|H\hat{T}_1|^2\lesssim\int_{y\leq 2B_0}|\Lambda Q|^2+\int_{B_0\leq y\leq 2B_0}\left|\frac{\log y}{y}\right|^2\lesssim|\log b|^2,$$

$$\int|H\hat{T}_2|^2\lesssim\int_{y\leq 2B_0}\left|\frac{y}{y^2b|\log b|}\right|^2\lesssim\frac{1}{b^2|\log b|},$$

and thus from (2.67), (3.26), (3.27):

$$\begin{aligned}\int|H\widehat{Mod}(t)|^2&\lesssim\left|\frac{\lambda_s}{\lambda}+b\right|^2\int|H\Lambda\hat{Q}_b|^2+|b_s+b^2|^2\int|H(\hat{T}_1+b\hat{T}_2)|^2\\&\lesssim\frac{b^4}{|\log b|^2}|\log b|^2\lesssim b^4|\log b|^2.\end{aligned}$$

Moreover, $\mathrm{Supp}(H\widehat{Mod})\subset[0,2B_0]$ and thus with (5.29):

$$|(\hat{\varepsilon}_2,H\widehat{Mod})|\lesssim\left(b^4|\log b|^2b^2|\log b|^2\right)^{\frac12}\lesssim b^3|\log b|^2. \tag{5.31}$$

We now claim the following bound for the small linear term, and the nonlinear term:

$$\int|H\hat{L}(\hat{\varepsilon})|^2+|H\hat{N}(\hat{\varepsilon})|^2\lesssim b^5. \tag{5.32}$$

Assume (5.32). Thus,

$$|(\hat{\varepsilon}_2,HL(\hat{\varepsilon}))|\lesssim b|\log b|^Cb^{\frac52}\lesssim b^3|\log b|^2. \tag{5.33}$$

(5.33), together with (5.27), (5.28), (5.30) and (5.31) concludes the proof of Proposition 5.3.

Proof of (5.32):

We recall that:

$$\hat{L}(\hat{\varepsilon}) = 3(\hat{Q}_b^2 - Q^2)\hat{\varepsilon}.$$

In the same way as previously, we let:

$$\hat{g} = 3(\hat{Q}_b^2 - Q^2),$$

for which we have the bounds:

$$\begin{aligned} |\hat{g}| + |\partial_y \hat{g}| + |H\hat{g}| &\lesssim b \quad \text{for} \quad y \le 1 \\ \left| \frac{\hat{g}}{1+|\log y|} \right| + |y\partial_y \hat{g}| + \left|y^2 H\hat{g}\right| &\lesssim \frac{b}{1+y^2} \quad \text{for} \quad y \ge 1. \end{aligned}$$

These bounds together with (2.69), (2.70), Lemma B.1 and the following decomposition:

$$H\hat{L} = \hat{\varepsilon} H\hat{g} - \hat{g}\Delta\hat{\varepsilon} - 2\partial_y \hat{g}\partial_y \hat{\varepsilon}$$

yield the expected bound for the linear term. We estimate afterwards the nonlinear term. We know that:

$$\hat{N}(\hat{\varepsilon}) = 3Q\hat{\varepsilon}^2 + 3\hat{f}\hat{\varepsilon}^2 + \hat{\varepsilon}^3.$$

where we note

$$\hat{f} = 3(\hat{Q}_b - Q)$$

and we are within the bounds of the profiles $(T_i)_{1\le i\le 3}$:

$$\begin{aligned} |\hat{f} + \partial_y \hat{f} + H\hat{f}| &\lesssim b \quad \text{for} \quad y \le 1 \\ \left| \frac{\hat{f}}{1+|\log y|} + y\partial_y \hat{f} + y^2 H\hat{f} \right| &\lesssim b \quad \text{for} \quad y \ge 1. \end{aligned}$$

With the estimates L^∞ of Lemma B.1, of the second localization (2.69) and (2.70), we have:

$$\begin{aligned} \int |H\hat{N}(\hat{\varepsilon})|^2 &\lesssim \frac{\Delta(\hat{\varepsilon}^2)^2}{1+y^2} + \frac{(\partial_y \hat{\varepsilon}^2)^2}{1+y^6} + \int \frac{\hat{\varepsilon}^4}{1+y^8} \\ &+ \|\hat{\varepsilon}\|_{L^\infty}^2 \int |H\hat{f}\hat{\varepsilon}|^2 + |\Delta\hat{\varepsilon}\hat{f}|^2 + |\partial_y \hat{f}\partial_y \hat{\varepsilon}|^2 \\ &+ \|\hat{\varepsilon}\|_{L^\infty}^4 \hat{\mathcal{E}}_2 + \|\hat{\varepsilon}\|_{L^\infty}^2 \int |\Delta\hat{\varepsilon}^2|^2 + \|\hat{\varepsilon}\|_{L^\infty}^2 \|\partial_y \hat{\varepsilon}\|_{L^\infty}^2 \int |\partial_y \hat{\varepsilon}|^2 \\ &\lesssim b^5. \end{aligned}$$

Proposition 5.3 is proved.

6. Proof of Proposition 4.2

6.1. **Improved bound.** The Lyapunov monotonicity property allow us to improve the a priori bounds under the a priori control (4.11) on the unstable direction.

Lemma 6.1 (Improved bounds under the a priori control (4.11))**.** *Assume that K in (4.7), (4.8), (4.9), and (4.10) has been chosen large enough. Then, $\forall t \in [0, t_1]$:*

$$0 \le b(t) \le 2b(0), \tag{6.1}$$

$$\int |\nabla\varepsilon|^2 \le \sqrt{b(0)}, \tag{6.2}$$

$$|\mathcal{E}_2(t)| \leq \frac{K}{2} b^2(t)|\log b(t)|^5, \tag{6.3}$$

$$|\mathcal{E}_4(t)| \leq \frac{K}{2}\frac{b^4(t)}{|\log b(t)|^2}. \tag{6.4}$$

Proof .
Step 1. Positivity and smallness of $b(t)$.

The proof of (6.1) is a direct consequence of the modulation equations. Indeed, this last equation (3.27) yields that

$$b_s \leq 0 \tag{6.5}$$

We must now prove that $b(t)$ can't be negative. We argue by contradiction. As $b(0) \geq 0$ and b(t) is a continuous function, we suppose that there exists t_0 such that $b(t_0) = 0$. With the modulation equations, we have that:

$$|b_s| \leq 2b^2 \tag{6.6}$$

Hence, there exists δ such that $b(t) = 0$ on $[t_0 - \delta, t_0]$, and thus from (4.10), $\lambda(t) = \lambda(t_0)$ and $u(t) = Q_{\lambda(t_0)}$ on $[t_0 - \delta, t_0]$. Iterating on $\delta > 0$, we conclude that u_0 is initially a rescaling of Q, a contradiction.

Step 2. Energy bound.

(6.2) is a consequence of the decrease of energy. Indeed, let

$$\tilde{\varepsilon} = \tilde{\alpha} + \varepsilon. \tag{6.7}$$

Then

$$\begin{aligned} E(u) &= \frac{1}{2}\int |\nabla u|^2 - \frac{1}{4}\int |u|^4 \\ &= \frac{1}{2}\left\{\int |\nabla Q|^2 + \int |\nabla\tilde{\varepsilon}|^2\right\} + \int \partial_y Q \partial_y \tilde{\varepsilon} - \frac{1}{4}\int \left[Q^4 + 4Q^3\tilde{\varepsilon} + 6Q^2\tilde{\varepsilon}^2 + 4Q\tilde{\varepsilon}^3 + \tilde{\varepsilon}^4\right] \\ &= E(Q) + (H\tilde{\varepsilon}, \tilde{\varepsilon}) - \frac{1}{4}\int \left[4Q\tilde{\varepsilon}^3 + \tilde{\varepsilon}^4\right]. \end{aligned} \tag{6.8}$$

Now,

$$\begin{aligned} (H\tilde{\varepsilon}, \tilde{\varepsilon}) &= (H\varepsilon, \varepsilon) + (H\tilde{\alpha}, \tilde{\alpha}) + 2(\tilde{\alpha}, H\varepsilon) \\ &= (H\varepsilon, \varepsilon) + O(b|\log b|^C). \end{aligned}$$

The last equality comes from Cauchy-Schwarz, the bound (4.9) for ε_2 and the inequalities:

$$\|\tilde{\alpha}\|_{L^2}^2 \lesssim |\log b|^C, \quad \|H\tilde{\alpha}\|_{L^2}^2 \lesssim b^4|\log b|^C.$$

Moreover, using (A.10), the orthogonality conditions (3.5), and the *a priori* bound on the unstable direction (4.11)

$$(H\varepsilon, \varepsilon) \geq c\int |\nabla\varepsilon|^2 - \frac{1}{c}\left[(\varepsilon, \Phi_M)^2 + (\varepsilon, \psi)^2\right] \gtrsim \int |\nabla\varepsilon|^2 + O\left(\frac{b^5}{|\log b|^2}\right).$$

Thus,

$$(H\tilde{\varepsilon}, \tilde{\varepsilon}) \gtrsim \int |\nabla\varepsilon|^2 + O(b|\log b|^C). \tag{6.9}$$

Let us see the nonlinear terms. We recall that:

$$\|\tilde{\varepsilon}\|_{L^\infty} \lesssim \|\varepsilon\|_{L^\infty} + \|\tilde{\alpha}\|_{L^\infty} \lesssim b|\log b| \quad \left\|\frac{\tilde{\alpha}}{y}\right\|_{L^2}^2 + \|\nabla\tilde{\alpha}\|_{L^2}^2 \lesssim b^2|\log b|^C.$$

Like this, with the bound (4.8) and the fact that:

$$\forall u \in H^1_{rad}(\mathbb{R}^4), \quad \int \frac{|u|^2}{y^2} \lesssim \int |\nabla u|^2, \quad \left(\int |u|^4\right)^{\frac14} \lesssim \left(\int |\nabla u|^2\right)^{\frac12}, \tag{6.10}$$

thus

$$\begin{aligned}\int \left[4Q\tilde{\varepsilon}^3 + \tilde{\varepsilon}^4\right] &\lesssim \|\tilde{\varepsilon}\|_{L^\infty}\left(\int \frac{|\varepsilon|^2}{1+y^2} + \left\|\frac{\tilde{\alpha}}{y}\right\|^2_{L^2}\right) + \left(\int |\nabla \varepsilon|^2\right)^2 + \left(\int |\nabla \tilde{\alpha}|^2\right)^2 \\ &\lesssim \sqrt{b(0)}\int |\nabla \varepsilon|^2 + O(b|\text{log}b|^C). \end{aligned} \tag{6.11}$$

The first inequality of (6.10) comes from Lemma (A.1). The second is a classical result of Sobolev, Gagliardo and Nirenberg in dimension $N = 4$. This results in general case and its proof is available in [2] with Theorem IX.9. By construction,

$$0 \le E(u) - E(Q) \lesssim b^2(0)|\text{log}b(0)|^C. \tag{6.12}$$

Injecting (6.9),(6.11) and (6.12) into (6.8) concludes the proof of (6.2).

Step 3. Control of $\mathcal{E}_4$.

We argue similarly as in [19]. $\forall t \in [0, t_1)$,

$$\begin{aligned}\mathcal{E}_4(t) &\le 2\left(\frac{\lambda(t)}{\lambda(0)}\right)^6\left[\mathcal{E}_4(0) + C\sqrt{b(0)}\frac{b^4(0)}{|\text{log}b(0)|^2}\right] + \frac{b^4(t)}{|\text{log}b(t)|^2} \\ &+ C\left[1 + \frac{K}{\text{log}M} + \sqrt{K}\right]\lambda^6(t)\int_0^t \frac{b}{\lambda^8}\frac{b^4}{|\text{log}b|^2}d\tau\end{aligned} \tag{6.13}$$

for some universal constant $C > 0$ independent of M.
Let us now consider two constants

$$\alpha_1 = 2 - \frac{C_1}{\sqrt{\text{log}M}}, \quad \alpha_2 = 2 + \frac{C_2}{\sqrt{\text{log}M}} \tag{6.14}$$

for some large enough universal constants C_1, C_2. We compute using the modulation equations (3.26), (3.27) and the bootstrap bound (4.10):

$$\begin{aligned}\frac{d}{ds}\left\{\frac{|\text{log}b|^{\alpha_i}b}{\lambda}\right\} &= \frac{|\text{log}b|^{\alpha_i}}{\lambda}\left[\left(1 - \frac{\alpha_i}{|\text{log}b|}\right)b_s - \frac{\lambda_s}{\lambda}b\right] \\ &= \frac{|\text{log}b|^{\alpha_i}}{\lambda}\left[\left(1 - \frac{\alpha_i}{|\text{log}b|}\right)b_s + b^2 + O\left(\frac{b^3}{|\text{log}b|}\right)\right] \\ &= \left(1 - \frac{\alpha_i}{|\text{log}b|}\right)\frac{|\text{log}b|^{\alpha_i}}{\lambda}\left[b_s + b^2\left(1 + \frac{\alpha_i}{|\text{log}b|} + O\left(\frac{1}{|\text{log}b|^2}\right)\right)\right] \\ &\begin{cases} \le 0 & \text{for } i = 1 \\ \ge 0 & \text{for } i = 2.\end{cases}\end{aligned}$$

Integrating this from 0 to t yields:

$$\frac{b(0)}{\lambda(0)}\left(\frac{|\text{log}b(0||}{|\text{log}b(t)|}\right)^{\alpha_2} \le \frac{b(t)}{\lambda(t)} \le \frac{b(0)}{\lambda(0)}\left(\frac{|\text{log}b(0||}{|\text{log}b(t)|}\right)^{\alpha_1}. \tag{6.15}$$

This yields in particular using the initial bound (4.3) and the bound (4.7):

$$\left(\frac{\lambda(t)}{\lambda(0)}\right)^6 \mathcal{E}_4(0) \le (b(t)|\text{log}b(t)|^{\alpha_2})^6\frac{\mathcal{E}_0}{(b(0)|\text{log}b(0)|^{\alpha_2})^6} \le \frac{b^4(t)}{|\text{log}b(t)|^2}, \tag{6.16}$$

$$\begin{aligned} C\left(\frac{\lambda(t)}{\lambda(0)}\right)^6\sqrt{b(0)}\frac{b^4(0)}{|\text{log}b(0)|^2} &\lesssim \left(\frac{b(t)|\text{log}b(t)|^{\alpha_2}}{b(0)|\text{log}b(0)|^{\alpha_2}}\right)^6\sqrt{b(0)}\frac{b^4(0)}{|\text{log}b(0)|^2}\\ &\lesssim C(b(t))^{4+\frac14}\leq \frac{b^4(t)}{|\text{log}b(t)|^2}. \end{aligned} \tag{6.17}$$

We now compute explicitly using $b=-\lambda\lambda_t+O\left(\frac{b^2}{|\text{log}b|}\right)$ from (3.26):

$$\begin{aligned} \int_0^t\frac{b}{\lambda^8}\frac{b^4}{|\text{log}b|^2}d\sigma &= \frac16\left[\frac{b^4}{\lambda^6|\text{log}b|^2}\right]_0^t-\frac16\int_0^t\frac{b_tb^3}{\lambda^6|\text{log}b|^2}\left(4+\frac{2}{|\text{log}b|}\right)d\tau\\ &+ O\left(\int_0^t\frac{b}{\lambda^8}\frac{b^5}{|\text{log}b|^2}d\tau\right) \end{aligned}$$

which implies using now $|b_s+b^2|\lesssim\frac{b^2}{|\text{log}b|}$ from (3.27) and (4.10):

$$\lambda^6(t)\int_0^t\frac{b}{\lambda^8}\frac{b^4}{|\text{log}b|^2}d\sigma\lesssim\left[1+O\left(\frac{1}{|\text{log}b_0|}\right)\right]\frac{b^4(t)}{|\text{log}b(t)|^2}.$$

Injecting this together with (6.16), (6.17) into (6.13) yields

$$\mathcal{E}_4(t)\leq C\frac{b^4(t)}{|\text{log}b(t)|^2}\left[1+\frac{K}{\text{log}M}+\sqrt{K}\right]$$

for some universal constant $C>0$ independent of K and M, and thus (6.4) follows for K large enough independent of M.

Step 4. Control of $\mathcal{E}_2$.
Similarly to the control of $\mathcal{E}_4$, we give the same proof as well as in [19]. We integrate the monotonicity formula (5.25) after recalling the estimate of the difference between $\hat{\mathcal{E}}_2$ and $\mathcal{E}_2$:

$$\begin{aligned} \mathcal{E}_2(t) &= \lambda^2(t)\|w_2(t)\|^2_{L^2}\lesssim\|H\zeta_b(t)\|^2_{L^2}+\lambda^2(t)\|\hat{w}_2(t)\|^2_{L^2} \\ &\lesssim b^4(t)|\text{log}b(t)|^2+\left(\frac{\lambda(t)}{\lambda(0)}\right)^2\left[\mathcal{E}_2(0)+b^2(0)|\text{log}b(0)|^2\right]+\lambda^2(t)\int_0^t\frac{b^3|\text{log}b|^2}{\lambda^4(\tau)}d\tau. \end{aligned} \tag{6.18}$$

From (4.3), (6.15):

$$\begin{aligned} \left(\frac{\lambda(t)}{\lambda(0)}\right)^2\left[\mathcal{E}_2(0)+b^2(0)|\text{log}b(0)|^2\right] &\lesssim \frac{(b(0))^{10}+b^2(0)|\text{log}b(0)|^2}{(b(0)|\text{log}b(0)|^{\alpha_2}|)^2}b^2(t)|\text{log}b(t)|^{2\alpha_2}\\ &\leq b^2(t)|\text{log}b(t)|^{4+\frac14}, \end{aligned}$$

We now use the bound $b_s\lesssim -b^2$ and (6.15) to estimate:

$$\begin{aligned} &\lambda^2(t)\int_0^t\frac{b^3|\text{log}b|^2}{\lambda^4(\tau)}d\tau\lesssim\lambda^2(t)\int_0^t\frac{-b_tb|\text{log}b|^2}{\lambda^2(\tau)}d\tau\\ \lesssim &\left(\frac{\lambda(t)}{\lambda(0)}\right)^2 b^2(0)|\text{log}b(0)|^{2\alpha_1}\int_0^t\frac{-b_t}{b|\text{log}b|^{2\alpha_1-2}}d\tau\\ \lesssim &\left(\frac{\lambda(t)}{\lambda(0)}\right)^2 b^2(0)|\text{log}b(0)|^{2\alpha_1}\frac{1}{|\text{log}b(0)|^{2\alpha_1-3}}\\ \lesssim & \ b^2(t)|\text{log}b(t)|^{2\alpha_2}\frac{|\text{log}b(0)|^3}{|\text{log}b(0)|^{2\alpha_2}}\lesssim b^2(t)|\text{log}b(t)|^{4+\frac14}. \end{aligned}$$

Injecting these bounds into (6.18) yields:

$$\mathcal{E}_2(t)\lesssim b^2(t)|\text{log}b(t)|^{4+\frac14}$$

and concludes the proof of (6.3).

6.2. **Dynamic of the unstable mode.** To conclude Proposition 4.2, we must study the dynamic of the unstable mode. We recall that $\kappa(t) = (\varepsilon(t), \psi)$.

Lemma 6.2 (Control of the unstable mode)**.** *There holds: for all* $t \in [0, T_1(a^+)]$,

$$\left|\frac{d\kappa}{ds} - \zeta\kappa\right| \le \sqrt{b}\frac{b^{\frac{5}{2}}}{|\log b|}. \tag{6.19}$$

Proof. We compute the equation satisfied by κ by taking the inner product of (3.8) with the well localized direction ψ to get:

$$\frac{d\kappa}{ds} - \zeta\kappa = E(\varepsilon) \tag{6.20}$$

with

$$E(\varepsilon) = (-\tilde{\Psi}_b, \psi) + (L(\varepsilon), \psi) + (N(\varepsilon), \psi) - (Mod, \psi) + \frac{\lambda_s}{\lambda}(\Lambda\varepsilon, \psi). \tag{6.21}$$

We now estimate all terms of RHS. We recall the exponential localization of ψ as well as the orthogonality $(\psi, \Lambda Q) = 0$. To begin, using (2.63)

$$\left|(\tilde{\Psi}_b, \psi)\right| = \left|\frac{1}{\zeta^2}(\tilde{\Psi}_b, H^2\psi)\right| = \left|\frac{1}{\zeta^2}(H^2\tilde{\Psi}_b, \psi)\right| \lesssim \left(\int |H^2(\tilde{\Psi}_b)|^2\right)^{\frac{1}{2}} \lesssim \sqrt{b}\frac{b^{\frac{5}{2}}}{|\log b|}. \tag{6.22}$$

From the definition (3.21) of $L(\varepsilon)$, we have the following bound:

$$|L(\varepsilon)| \lesssim by^{10}|\varepsilon|.$$

Thus,

$$|(L(\varepsilon), \psi)| \lesssim b\left|\left(\frac{|\varepsilon|}{y^2(1+y^2)(1+|\log y|)}, (1+y^{14})(1+|\log y|)\psi\right)\right| \lesssim \sqrt{b}\frac{b^{\frac{5}{2}}}{|\log b|}. \tag{6.23}$$

In the same way, with (3.11),

$$|N(\varepsilon)| \lesssim \left(by^{10}|\varepsilon| + \left\|\frac{\varepsilon}{1+y}\right\|_{L^\infty}^2 y^2\right)|\varepsilon|.$$

(B.9) and the fact that $\forall i, \left\|y^i\psi\right\|_{L^\infty} \lesssim 1$ yield:

$$|(N(\varepsilon), \psi)| \lesssim \sqrt{b}\frac{b^{\frac{5}{2}}}{|\log b|}. \tag{6.24}$$

With the notation of Lemma 2.3, we have

$$\begin{aligned} |(Mod, \psi)| &= \left|\left(-\left(\frac{\lambda_s}{\lambda} + b\right)\Lambda\tilde{Q}_b + (b_s + b^2)(\tilde{T}_1 + 2b\tilde{T}_2), \psi\right)\right| \\ &\lesssim \left|\frac{\lambda_s}{\lambda} + b\right| |(\Lambda Q + \Lambda\tilde{\alpha}, \psi)| + \left|b_s + b^2\right| \left|\left(\tilde{T}_1 + 2b\tilde{T}_2, \psi\right)\right|. \end{aligned}$$

But

$$|\Lambda\tilde{\alpha}| + \left|2b\tilde{T}_2\right| \lesssim by^{10}$$

and

$$\left|\left(\tilde{T}_1, \psi\right)\right| = \left|\frac{-1}{\zeta}\left(\tilde{T}_1, H\psi\right)\right| = \left|\frac{-1}{\zeta}\left(H\tilde{T}_1, \psi\right)\right| \lesssim \left|\int_{y \ge B_1}\left(H\tilde{T}_1 - \Lambda Q\right)\psi\right| \lesssim b.$$

Hence, with the modulation equations

$$|(Mod,\psi)|\lesssim\sqrt{b}\frac{b^{\frac52}}{|\log b|}. \tag{6.25}$$

For the last term, we use (B.1).

$$\left|\frac{\lambda_s}{\lambda}(\Lambda\varepsilon,\psi)\right|=\left|\frac{\lambda_s}{\lambda}\right|\left|\left(\frac{\Lambda\varepsilon}{y^2(1+y^2)(1+|\log y|)},y^2(1+y^2)(1+|\log y|)\psi\right)\right|\lesssim\sqrt{b}\frac{b^{\frac52}}{|\log b|}. \tag{6.26}$$

This concludes the proof of Lemma 6.2.

6.3. **Conclusion.** We have at our disposal all the elements to finish the proof using a rough argument which does not give a sharp information on the link between the initial data and the choice of a^+, and uniqueness for example is not covered at this stage. The keystone of the proof is the fact that the map:

$$\begin{array}{rcl}\left[-2\frac{b_0^{\frac52}}{|\log b_0|};2\frac{b_0^{\frac52}}{|\log b_0|}\right] & \to & \mathbb{R}^+\\ a^+ & \to & T_1(a^+)\end{array}$$

is continuous as a consequence of the strictly outgoing behavior on exit (6.20) defined by (4.11). This classical argument is displayed in detail in [3], Lemma 6, in a more complicated setting and therefore left to the reader. Hence, we also have the continuity of the map

$$\begin{array}{rcl}\left[-2\frac{b_0^{\frac52}}{|\log b_0|};2\frac{b_0^{\frac52}}{|\log b_0|}\right] & \to & \mathbb{R}^+\\ a^+ & \to & \kappa(T_1(a^+)).\end{array}$$

In agreement with the dynamics of the unstable mode found in the last subsection, we know that, for $a^+=2\frac{b_0^{\frac52}}{|\log b_0|}$:

$$\frac{d}{ds}\kappa(0)=2\zeta\frac{b_0^{\frac52}}{|\log b_0|}+O\left(\frac{b_0^3}{|\log b_0|}\right)>0$$

and then

$$\kappa\left(T_1\left(2\frac{b_0^{\frac52}}{|\log b_0|}\right)\right)=2\frac{b_0^{\frac52}}{|\log b_0|}. \tag{6.27}$$

Likewise,

$$\kappa\left(T_1\left(-2\frac{b_0^{\frac52}}{|\log b_0|}\right)\right)=-2\frac{b_0^{\frac52}}{|\log b_0|}. \tag{6.28}$$

By continuity, there exists $a^+\in\left]-2\frac{b_0^{\frac52}}{|\log b_0|};2\frac{b_0^{\frac52}}{|\log b_0|}\right[$ such that

$$\kappa\left(T_1\left(a^+\right)\right)=0. \tag{6.29}$$

In addition, according to the definition of an exit time and Lemma 6.1, we have two choices. Either $|\kappa(T_1(a^+))|=2\frac{b(T_1(a^+))^{\frac52}}{|\log b(T_1(a^+))|}$ or $T_1(a^+)$ is the life time of the solution.

If the first possibility is the good one, the condition (6.29) gives

$$2\frac{b(T_1(a^+))^{\frac{5}{2}}}{|\log b(T_1(a^+))|} = 0.$$

As we have proved that $b(t) > 0$ for $t < T$, where T is the life time of the solution, we have thus the second possibility. Notice that in this case, the two choices tally. This is exactly Proposition 4.2.

Appendix A. L^2 coercivity estimates

In this appendix, we at first prove the Hardy inequalities for functions $u \in H^2_{rad}(\mathbb{R}^4)$. We will use afterwards this results to establish properties of weighted sub-coercivity for H and H^2, which allows us to obtain coercive estimates for these operators under additional orthogonality conditions. These coercivity estimates are crucial in our study. The proof lies in the continuation of the analysis in [5].

A.1. **Hardy inequalities.**

Lemma A.1. *There exists a constant C for which there holds, for any $v \in H^1_{rad}(\mathbb{R}^4)$*

$$\left[\int_{\mathbb{R}^4} \frac{v(y)^2}{y^2}\right]^{\frac{1}{2}} + \sup_{y\in\mathbb{R}^4} (|yv(y)|) \leq C\left[\int_{\mathbb{R}^4} |\nabla v(y)|^2\right]^{\frac{1}{2}}, \tag{A.1}$$

$$\|v\|^2_{L^\infty_{y\geq 1}} \lesssim \int_{y\geq 1}\left(\frac{|\nabla v|^2}{y^2}+\frac{|v|^2}{y^4}\right) + \int_{\frac{1}{2}\leq y\leq 1}\frac{|v|^2}{y^2} \tag{A.2}$$

Proof . By integration-by-parts:

$$\begin{aligned}\int \frac{v(y)^2}{y^2} &= \left[\frac{v(y)^2y^2}{2}\right]_0^\infty - \int \frac{v(y)\partial_y v(y)}{y}\\ &\lesssim \left[\int \frac{v(y)^2}{y^2}\right]^{\frac{1}{2}}\left[\int |\nabla v(y)|^2\right]^{\frac{1}{2}}.\end{aligned}$$

Next:

$$|v(y)| \lesssim \int_y^{+\infty} \frac{|\partial_y v|}{y^3} \lesssim \frac{1}{y}\left(\int |\partial_y v|^2\right)^{\frac{1}{2}},$$

$$|v^2(y)| \lesssim \int_{1\leq y\leq 2} |v|^2 + \int_{y\geq 1}\frac{|v||\partial_y v|}{y^3} \lesssim \left(\int_{y\geq 1}\frac{|\partial_y v|^2}{y^2}\right)^{\frac{1}{2}}\left(\int \frac{|v|^2}{y^4}\right)^{\frac{1}{2}}.$$

This concludes the proof of Lemma A.1.

Lemma A.2 (Hardy inequalities)**.** *$\forall R > 2$, $\forall v \in H^2_{rad}(\mathbb{R}^4)$, $\forall \gamma > 0$ there holds the following controls:*

$$\int |\partial_{yy}v|^2 + \int \frac{|\partial_y v|^2}{y^2} \lesssim \int (\Delta v)^2, \tag{A.3}$$

$$\int_{y\leq R} \frac{|v|^2}{y^4(1+|\log y|)^2} \lesssim \int_{y\leq R}\frac{|\partial_y v|^2}{y^2} + \int_{1\leq y\leq 2}|v|^2, \tag{A.4}$$

$$\int_{1\leq y\leq R} \frac{|v|^2}{y^{4+\gamma}(1+|\log y|)^2} \lesssim \int_{1\leq y\leq R}\frac{|\nabla v|^2}{y^{2+\gamma}(1+|\log y|)^2} + C_\gamma\int_{1\leq y\leq 2}|v|^2 \tag{A.5}$$

Proof. Let v be smooth and radially symmetric. (A.3) follows from the explicit formula after integration by parts

$$\int (\Delta v)^2 = \int \left(\partial_{yy} v + \frac{3}{y}\partial_y v\right)^2 = \int |\partial_{yy} v|^2 + 3\int \frac{|\partial_y v|^2}{y^2}.$$

To prove (A.4) and (A.5), from the one dimensional Sobolev embedding $H^1(1 \leq y \leq 2)$ in $L^\infty(1 \leq y \leq 2)$, we obtain

$$|v(1)|^2 \lesssim \int_{1\leq y\leq 2} \left(|v|^2 + |\partial_y v|^2\right). \tag{A.6}$$

Let $f(y) = -\frac{\mathbf{e}_y}{y^3(1+\log y)}$ so that $\nabla . f = \frac{1}{y^4(1+\log y)^2}$, and integrate by parts to get:

$$\begin{aligned} \int_{1\leq y\leq R} \frac{|v|^2}{y^4(1+|\log y|)^2} &= \int_{1\leq y\leq R} |v|^2 \nabla . f \\ &= -\left[\frac{|v|^2}{1+\log y}\right]_1^R + 2\int_{y\leq R} \frac{v\partial_y v}{y^3(1+\log y)} \\ &\lesssim |v(1)|^2 + \left(\int_{y\leq R} \frac{|v|^2}{y^4(1+|\log y|)^2}\right)^{\frac{1}{2}} \left(\int_{y\leq R} \frac{|\partial_y v|^2}{y^2}\right)^{\frac{1}{2}}. \end{aligned} \tag{A.7}$$

Similarly, using $\tilde{f}(y) = -\frac{\mathbf{e}_y}{y^3(1-\log y)}$, we get:

$$\begin{aligned} \int_{\varepsilon\leq y\leq 1} \frac{|v|^2}{y^4(1-\log y)^2} &= \int_{\varepsilon\leq y\leq 1} |v|^2 \nabla . f \\ &= \left[\frac{|v|^2}{1-\log y}\right]_\varepsilon^1 + 2\int_{y\leq 1} \frac{v\partial_y v}{y^3(1-\log y)} \\ &\lesssim |v(1)|^2 + \left(\int_{y\leq R} \frac{|v|^2}{y^4(1+|\log y|)^2}\right)^{\frac{1}{2}} \left(\int_{y\leq R} \frac{|\partial_y v|^2}{y^2}\right)^{\frac{1}{2}}. \end{aligned} \tag{A.8}$$

(A.6), (A.7) and (A.8) now yield (A.4). To prove (A.5), let $\gamma > 0$, and

$$f(y) = -\frac{\mathbf{e}_y}{y^{\gamma+3}(1+\log y)^2}$$

so that for $y \geq 1$

$$\nabla . f(y) = \frac{1}{y^{\gamma+4}(1+\log y)^2}\left[\gamma + \frac{2}{1+\log y}\right] \geq \frac{\gamma}{y^{\gamma+4}(1+\log y)^2}.$$

We then integrate by parts to get:

$$\begin{aligned} \gamma\int_{1\leq y\leq R} \frac{|v|^2}{y^{4+\gamma}(1+|\log y|)^2} &\leq \int_{1\leq y\leq R} |v|^2 \nabla . f \\ &\leq -\left[\frac{|v|^2}{y^{3+\gamma}(1+|\log y|)^2}\right]_1^R + 2\int_{1\leq y\leq R} \frac{|v\partial_y v|}{y^{\gamma+3}(1+\log y)^2} \\ &\leq C\int_{1\leq y\leq 2} |v|^2 + 2\left(\int_{y\leq R} \frac{|v|^2}{y^{4+\gamma}(1+|\log y|)^2}\right)^{\frac{1}{2}} \left(\int_{y\leq R} \frac{|\nabla v|^2}{y^{2+\gamma}(1+|\log y|)^2}\right)^{\frac{1}{2}}. \end{aligned}$$

and (A.5) follows.

A.2. **Sub-positive estimates for H.** The following lemma highlights the negative part of the operator H. We recall that this operator possesses a unique non-positive direction ψ.

Lemma A.3. *Let $u \in H^2_{rad}(\mathbb{R}^4)$, then there exists a constant $C > 0$ such that:*

$$(Hu, u) \geq -C\,(u, \psi)^2\,. \tag{A.9}$$

Proof. Let $u \in H^2_{rad}(\mathbb{R}^4)$. There exists an unique decomposition of u:

$$u = \kappa\psi + v$$

with the orthogonality condition

$$(\psi, v) = 0.$$

By definition, we have

$$\kappa = \frac{(u, \psi)}{(\psi, \psi)}.$$

Moreover, the uniqueness of the negative direction of H gives

$$(Hv, v) \geq 0.$$

Thus,

$$\begin{aligned}
(Hu, u) &= \kappa^2(H\psi, \psi) + \kappa\left[(Hv, \psi) + (v, H\psi)\right] + (Hv, v) \\
&= \kappa^2(H\psi, \psi) - 2\zeta\kappa(v, \psi) + (Hv, v) \\
&\geq \frac{(H\psi, \psi)}{(\psi, \psi)^2}\,(u, \psi)^2 \\
&\geq \frac{-\zeta}{(\psi, \psi)}\,(u, \psi)^2\,.
\end{aligned}$$

A.3. **Sub-coercivity estimates.** In this subsection, we prove sub-coercivity estimates for H and H^2 which are the key to the proof of coercive estimates for these operators under additional orthogonality conditions.

Lemma A.4 (Sub-coercivity estimates for H)**.** *Let $u \in H^2_{rad}(\mathbb{R}^4)$, then there exist constants $\delta > 0, C > 0$ such that:*

$$(H\varepsilon, \varepsilon) \geq c\int |\nabla\varepsilon|^2 - \frac{1}{c}\left[(\varepsilon, \Phi_M)^2 + (\varepsilon, \psi)^2\right], \tag{A.10}$$

$$\int |\partial_{yy}u|^2 + \int \frac{|\partial_y u|^2}{y^2} + \int \frac{u^2}{y^4(1+|\mathrm{log} y|)^2} - C\left[\int \frac{|\partial_y u|^2}{1+y^4} + \int \frac{u^2}{1+y^8}\right] \lesssim \int (Hu)^2. \tag{A.11}$$

Proof. (A.11) is a direct consequence of the inequalities (A.3), (A.4) and the following decomposition:

$$\begin{aligned}
\int (Hu)^2 &= \int (\Delta u + Vu)^2 = \int (\Delta u)^2 - 2\int V(\partial_y u)^2 + \int (\Delta V + V^2)u^2 \\
&\gtrsim \left[\int (\Delta u)^2 + \int \frac{u^2}{1+y^6}\right] - C\left[\int \frac{|\partial_y u|^2}{1+y^4} + \int \frac{u^2}{1+y^8}\right]
\end{aligned}$$

where we used that

$$V(y) = \frac{192}{1+y^4}\left[1 + O\left(\frac{1}{1+y^2}\right)\right] \quad \text{as} \quad y \to +\infty.$$

Lemma A.5 (Weighted sub-coercivity for H)**.** *Let $u\in H^4_{rad}(\mathbb{R}^4)$, then there exists a constant C such that:*

$$\begin{aligned}
&\int\frac{|u|^2}{y^4(1+y^4)(1+|\log y|)^2}+\int\frac{|\partial_y u|^2}{y^6(1+|\log y|)^2}+\int\frac{|\partial_{yy}u|^2}{y^4(1+|\log y|)^2}\\
+&\int\frac{|\partial_y^3 u|^2}{y^2(1+|\log y|)^2}+\int\frac{|\partial_y^4 u|^2}{(1+|\log y|)^2}\\
-&C\left[\int\frac{|u|^2}{y^2(1+y^8)(1+|\log y|)^2}+\int\frac{|\partial_y u|^2}{y^4(1+y^4)(1+|\log y|)^2}\right]\\
\lesssim&\int\frac{|Hu|^2}{y^4(1+|\log y|)^2}+\int\frac{|\partial_y Hu|^2}{y^2(1+|\log y|)^2}+\int\frac{|\partial_{yy}Hu|^2}{(1+|\log y|)^2}. \qquad (A.12)
\end{aligned}$$

Remark A.6. Using (A.1), (A.4), and (A.5), $u\in H^4_{rad}(\mathbb{R}^4)$ yields

$$\begin{aligned}
&\int\frac{|u|^2}{y^4(1+y^4)(1+|\log y|)^2}+\int\frac{|\partial_y u|^2}{y^6(1+|\log y|)^2}+\int\frac{|\partial_{yy}u|^2}{y^4(1+|\log y|)^2}\\
+&\int\frac{|\partial_y^3 u|^2}{y^2(1+|\log y|)^2}+\int\frac{|\partial_y^4 u|^2}{(1+|\log y|)^2}\\
+&\int\frac{|Hu|^2}{y^4(1+|\log y|)^2}+\int\frac{|\partial_y Hu|^2}{y^2(1+|\log y|)^2}+\int\frac{|\partial_{yy}Hu|^2}{(1+|\log y|)^2}<\infty.
\end{aligned}$$

Proof of Lemma A.5: Let $\chi(y)$ be a smooth cut-off function with support in $y\geq 1$ and equal to 1 for $y\geq 2$.

$$\begin{aligned}
&\int\chi\frac{|Hu|^2}{y^4(1+|\log y|)^2}=\int\chi\frac{\left[-\partial_y\left(y^3\partial_y u\right)-y^3Vu\right]^2}{y^{10}(1+|\log y|)^2}\\
=&\int\chi\frac{\left|\partial_y\left(y^3\partial_y u\right)\right|^2}{y^{10}(1+|\log y|)^2}+2\int\chi\frac{\partial_y\left(y^3\partial_y u\right)Vu}{y^7(1+|\log y|)^2}+\int\chi\frac{V^2u^2}{y^4(1+|\log y|)^2}\\
=&\int\chi\frac{\left|\partial_y\left(y^3\partial_y u\right)\right|^2}{y^{10}(1+|\log y|)^2}-2\int\chi\frac{V(\partial_y u)^2}{y^4(1+|\log y|)^2}+\int\chi\frac{V^2u^2}{y^4(1+|\log y|)^2}\\
+&\int|u|^2\Delta\left(\chi\frac{V}{y^4(1+|\log y|)^2}\right).
\end{aligned}$$

We now observe that for $k\geq 0$

$$|\partial_y^k V(y)|\lesssim\frac{1}{1+y^{4+k}}$$

and thus,

$$\begin{aligned}
&\left|-2\int\chi\frac{V(\partial_y u)^2}{y^4(1+|\log y|)^2}+\int\chi\frac{V^2u^2}{y^4(1+|\log y|)^2}+\int\chi u^2\Delta\left(\frac{V}{y^4(1+|\log y|)^2}\right)\right|\\
\lesssim&\int\frac{|u|^2}{y^2(1+y^8)(1+|\log y|)^2}+\int\frac{|\partial_y u|^2}{y^4(1+y^4)(1+|\log y|)^2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int\chi\frac{|Hu|^2}{y^4(1+|\log y|)^2}\gtrsim&\int\chi\frac{\partial_y\left(y^3\partial_y u\right)^2}{y^{10}(1+|\log y|)^2}\\
&-C\left[\int\frac{|u|^2}{y^2(1+y^8)(1+|\log y|)^2}+\int\frac{|\partial_y u|^2}{y^4(1+y^4)(1+|\log y|)^2}\right].
\end{aligned}$$

We may apply twice the Hardy inequality (A.5) with $\gamma = 8$ and $\gamma = 4$ and get for a sufficiently large universal constant R:

$$\begin{aligned}\int \chi \frac{\partial_y \left(y^3 \partial_y u\right)^2}{y^{10}(1+|\log y|)^2} &\gtrsim \int_{y\geq R} \frac{|\partial_y u|^2}{y^6(1+|\log y|)^2} - C\int \frac{|\partial_y u|^2}{y^4(1+y^4)(1+|\log y|)^2}\\ &\gtrsim \int_{y\geq R} \frac{|u|^2}{y^4(1+y^4)(1+|\log y|)^2}\\ &- C\left[\int \frac{|u|^2}{y^2(1+y^8)(1+|\log y|)^2} + \int \frac{|\partial_y u|^2}{y^4(1+y^4)(1+|\log y|)^2}\right].\end{aligned}$$

and finally,

$$\begin{aligned}&\int_{y\geq 2} \frac{|u|^2}{y^4(1+y^4)(1+|\log y|)^2} + \int_{y\geq 2} \frac{|\partial_y u|^2}{y^6(1+|\log y|)^2} + \int_{y\geq 2} \frac{|\partial_{yy} u|^2}{y^4(1+|\log y|)^2}\\ &- C\left[\int \frac{|u|^2}{y^2(1+y^6)(1+|\log y|)^2} + \int \frac{|\partial_y u|^2}{y^4(1+y^4)(1+|\log y|)^2}\right]\\ &\lesssim \int \chi \frac{|Hu|^2}{y^4(1+|\log y|)^2}. \qquad \text{(A.13)}\end{aligned}$$

Now, the control of the third derivative for $y \leq 1$ follows from:

$$\begin{aligned}\int \chi \frac{|\partial_y Hu|^2}{y^2(1+|\log y|)^2} \geq \int \chi \frac{|\partial_y(\Delta u + Vu)|^2}{y^2(1+|\log y|)^2} \gtrsim \int \chi \frac{|\partial_y^3 u|^2}{y^2(1+|\log y|)^2}\\ -C\left[\int \chi \frac{|u|^2}{(1+y^{12})(1+|\log y|)^2} + \int \chi \frac{|\partial_y u|^2}{y^6(1+|\log y|)^2} + \int \chi \frac{|\partial_{yy} u|^2}{y^4(1+|\log y|)^2}\right] \qquad \text{(A.14)}\end{aligned}$$

and the fourth derivative from:

$$\begin{aligned}&\int \chi \frac{|\partial_{yy} Hu|^2}{(1+|\log y|)^2} \geq \int \chi \frac{|\partial_{yy}(\Delta u + Vu)|^2}{(1+|\log y|)^2}\\ &\gtrsim \int \chi \frac{|\partial_y^4 u|^2}{(1+|\log y|)^2} - C\left[\int \chi \frac{|\partial_y^3 u|^2}{y^2(1+|\log y|)^2} + \int \chi \frac{|\partial_{yy} u|^2}{y^4(1+|\log y|)^2}\right] \qquad \text{(A.15)}\\ &- C\left[\int \chi \frac{|u|^2}{(1+y^{12})(1+|\log y|)^2} + \int \chi \frac{|\partial_y u|^2}{y^6(1+|\log y|)^2}\right]. \qquad \text{(A.16)}\end{aligned}$$

(A.13),(A.14) and (A.16) yield (A.12), away from the origin. Let us study this control near the origin. Let $\zeta = (1-\chi)^{\frac{1}{2}}$. With Lemma A.4, we have that:

$$\begin{aligned}&\int \zeta^2 \frac{|Hu|^2}{y^4(1+|\log y|^2)} \gtrsim \int \zeta^2 |Hu|^2 \gtrsim |H\zeta u|^2 - C\int_{1\leq y\leq 2} \left(|\partial_y u|^2 + |u|^2\right)\\ &\gtrsim \int \frac{|\zeta u|^2}{y^4(1+|\log y|^2)} - C\int \frac{|\zeta u|^2}{y^2} \gtrsim \int_{y\leq 1} \frac{|u|^2}{y^4(1+|\log y|^2)} - C\int_{y\leq 2} \frac{|u|^2}{y^2}.\end{aligned}$$

Now, by definition, we have:

$$Hu = -\frac{1}{y^3}\frac{\partial}{\partial_y}\left(y^3 \partial_y u\right) - Vu.$$

Hence

$$\partial_y u = -\frac{1}{y^3}\int_{\tau\leq y} (Vu + Hu). \qquad \text{(A.17)}$$

We then estimate from Cauchy-Schwarz and Fubini:

$$\begin{aligned}\int_{y\le 1}\frac{|\partial_y u|^2}{y^6(1+|\mathrm{log}y|^2)} &= \int_{y\le 1}\frac{|\partial_y u|^2}{y^{12}(1+|\mathrm{log}y|^2)}\left(\int_0^y Vu+Hu\right)^2\\ &\lesssim \int_{y\le 1}\frac{|\partial_y u|^2}{y^2(1+|\mathrm{log}y|^2)}\left(\int_0^y\frac{|Vu(\tau)|^2+|Hu(\tau)|^2}{\tau^3}\right)\\ &\lesssim \int_{\tau\le 1}\frac{|Vu(\tau)|^2+|Hu(\tau)|^2}{\tau^3}\left(\int_{\tau\le y\le 1}\frac{1}{y^2(1+|\mathrm{log}y|^2)}\right)\\ &\lesssim \int_{y\le 1}\frac{|Hu|^2}{y^4(1+|\mathrm{log}y|)^2}. \end{aligned}\tag{A.18}$$

Finally, for the control of the other derivatives near the origin:

$$\partial_{yy}u=-Hu-3\frac{\partial_y u}{y}-Vu.$$

Thus,

$$\int_{y\le 1}\frac{|\partial_{yy}u|^2}{y^4(1+|\mathrm{log}y|^2)}\lesssim\int_{y\le 1}\frac{|Hu|^2}{y^4(1+|\mathrm{log}y|^2)}+\int_{y\le 1}\frac{|\partial_y u|^2}{y^6(1+|\mathrm{log}y|^2)}+\int_{y\le 1}\frac{|u|^2}{y^4(1+|\mathrm{log}y|^2)},\tag{A.19}$$

$$\begin{aligned}\int_{y\le 1}\frac{|\partial_y^3 u|^2}{y^2(1+|\mathrm{log}y|^2)} &\lesssim \int_{y\le 1}\frac{|\partial_y Hu|^2}{y^2(1+|\mathrm{log}y|^2)}+\int_{y\le 1}\frac{|\partial_{yy}u|^2}{y^4(1+|\mathrm{log}y|^2)}\\ &+ \int_{y\le 1}\frac{|\partial_y u|^2}{y^6(1+|\mathrm{log}y|^2)}+\int_{y\le 1}\frac{|u|^2}{y^4(1+|\mathrm{log}y|^2)},\end{aligned}\tag{A.20}$$

$$\begin{aligned}\int_{y\le 1}\frac{|\partial_y^4 u|^2}{(1+|\mathrm{log}y|^2)} &\lesssim \int_{y\le 1}\frac{|\partial_{yy}Hu|^2}{(1+|\mathrm{log}y|^2)}+\int_{y\le 1}\frac{|\partial_y^3 u|^2}{y^4(1+|\mathrm{log}y|^2)}+\int_{y\le 1}\frac{|\partial_{yy}u|^2}{y^4(1+|\mathrm{log}y|^2)}\\ &+ \int_{y\le 1}\frac{|\partial_y u|^2}{y^6(1+|\mathrm{log}y|^2)}+\int_{y\le 1}\frac{|u|^2}{y^4(1+|\mathrm{log}y|^2)}.\end{aligned}\tag{A.21}$$

This concludes the proof.

We now combine the results of Lemma A.4 and Lemma A.5

Lemma A.7 (Sub-coercivity for H^2)**.** *Let $u\in H^4_{rad}(\mathbb{R}^4)$. Then,*

$$\begin{aligned}\int|H^2u|^2 &\gtrsim \int\frac{|Hu|^2}{y^4(1+|\mathrm{log}y|)^2}+\int\frac{|\partial_y Hu|^2}{y^2(1+|\mathrm{log}y|)^2}+\int\left|\frac{\partial_{yy}Hu}{(1+|\mathrm{log}y|)^2}\right|^2 \qquad (A.22)\\ &+ \int\frac{|u|^2}{y^4(1+y^4)(1+|\mathrm{log}y|)^2}+\int\frac{|\partial_y u|^2}{y^6(1+|\mathrm{log}y|)^2}+\int\frac{|\partial_{yy}u|^2}{y^4(1+|\mathrm{log}y|)^2}\\ &+ \int\frac{|\partial_y^3 u|^2}{y^2(1+|\mathrm{log}y|)^2}\int\frac{|\partial_y^4 u|^2}{(1+|\mathrm{log}y|)^2}-C\int\frac{|u|^2}{y^2(1+y^8)(1+|\mathrm{log}y|)^2}\\ &- C\left[\int\frac{|\partial_y u|^2}{y^4(1+y^4)(1+|\mathrm{log}y|)^2}+\int\frac{|\partial_y Hu|^2}{1+y^4}+\int\frac{Hu^2}{1+y^8}\right].\end{aligned}$$

A.4. **Coercivity of H^2.** We are now in position to derive the fundamental coercivity property of H^2 at the heart of our analysis.

Lemma A.8 (Coercivity of H^2)**.** *Let $M\ge 1$ be a large enough universal constant. Let Φ_M be given by (3.2). Then there exists a universal constant $C(M)>0$ such that for all $u\in H^4_{rad}(\mathbb{R}^4)$ satisfying the orthogonality conditions:*

$$(u,\Phi_M)=0,\qquad (Hu,\Phi_M)=0$$

there holds:

$$\begin{aligned} &\int \frac{|Hu|^2}{y^4(1+|\log y|)^2} + \int \frac{|\partial_y Hu|^2}{y^2(1+|\log y|)^2} + \int \frac{|\partial_{yy} Hu|^2}{(1+|\log y|)^2} \\ + \quad &\int \frac{|u|^2}{y^4(1+y^4)(1+|\log y|)^2} \int \frac{|\partial_y u|^2}{y^6(1+|\log y|)^2} + \int \frac{|\partial_{yy} u|^2}{y^4(1+|\log y|)^2} \\ + \quad &\int \frac{|\partial_y^3 u|^2}{y^2(1+|\log y|)^2} + \int \frac{|\partial_y^4 u|^2}{(1+|\log y|)^2} \le C(M) \int |H^2(u)|^2. \end{aligned} \tag{A.23}$$

Proof. We argue by contradiction. Let $M > 0$ be fixed and consider a normalized sequence u_n

$$\begin{aligned} &\int \frac{|Hu_n|^2}{y^4(1+|\log y|)^2} + \int \frac{|\partial_y Hu_n|^2}{y^2(1+|\log y|)^2} + \int \frac{|\partial_{yy} Hu_n|^2}{(1+|\log y|)^2} \\ + \quad &\int \frac{|u_n|^2}{y^4(1+y^4)(1+|\log y|)^2} + \int \frac{|\partial_y u_n|^2}{y^6(1+|\log y|)^2} + \int \frac{|\partial_{yy} u_n|^2}{y^4(1+|\log y|)^2} \\ + \quad &\int \frac{|\partial_y^3 u_n|^2}{y^2(1+|\log y|)^2} + \int \frac{|\partial_y^4 u_n|^2}{(1+|\log y|)^2} = 1 \end{aligned} \tag{A.24}$$

satisfying the orthogonality conditions

$$(u_n, \Phi_M) = 0, \qquad (Hu_n, \Phi_M) = 0$$

and

$$\int |H^2(u_n)|^2 \le \frac{1}{n}. \tag{A.25}$$

The normalization condition implies that the sequence u_n is uniformly bounded in H^4_{loc}. As a consequence, we can assume that u_n weakly converges in H^4_{loc} to u_∞. Moreover, u_∞ satisfies the equation

$$H^2 u_\infty = 0 \ \text{ for } \ r > 0.$$

Integrating this ODE leads to

$$Hu_\infty = \alpha\Lambda Q + \beta\Gamma \ \text{ for } \ r > 0.$$

Using the condition $u_\infty \in H^4_{loc}$, we can determine that $\beta = 0$. Hence, the function u_∞ can be written in the form

$$u_\infty = -\alpha T_1 + \gamma\Lambda Q + \delta\Gamma.$$

The condition $u_\infty \in H^4_{loc}$ yields that $\delta = 0$. Passing trough the limit in the orthogonality conditions, using that u_n converges to u_∞ weakly in H^4_{loc}, we conclude that u_∞ satisfies

$$(u_\infty, \Phi_M) = 0, \qquad (Hu_\infty, \Phi_M) = 0.$$

We may therefore determine the constants α and γ using (3.2), (3.5) which yield $\alpha = \gamma = 0$ and thus $u_\infty = 0$.

The sub-coercivity bound (A.22) together with (A.25) ensures:

$$\begin{aligned}\frac{1}{n} &\gtrsim \int |H^2(u_n)|^2 \gtrsim \int \frac{|Hu_n|^2}{y^4(1+|\log y|)^2} + \int \frac{|\partial_y Hu_n|^2}{y^2(1+|\log y|)^2} + \int \frac{|\partial_{yy} Hu_n|^2}{(1+|\log y|)^2}\\ &+ \int \frac{|u_n|^2}{y^4(1+y^4)(1+|\log y|)^2} + \int \frac{|\partial_y u_n|^2}{y^6(1+|\log y|)^2} + \int \frac{|\partial_{yy} u_n|^2}{y^4(1+|\log y|)^2}\\ &+ \int \frac{|\partial_y^3 u_n|^2}{y^2(1+|\log y|)^2} + \int \frac{|\partial_y^4 u_n|^2}{(1+|\log y|)^2} - C\left[\int \frac{|\partial_y Hu_n|^2}{1+y^4} + \int \frac{Hu_n^2}{1+y^8}\right]\\ &- C\left[\int \frac{|u_n|^2}{y^2(1+y^8)(1+|\log y|)^2} + \int \frac{|\partial_y u_n|^2}{y^4(1+y^4)(1+|\log y|)^2}\right].\end{aligned}$$

Coupling this with the normalization condition we obtain that

$$\int \frac{|\partial_y Hu_n|^2}{1+y^4} + \int \frac{Hu_n^2}{1+y^8} + \int \frac{|u_n|^2}{y^2(1+y^8)(1+|\log y|)^2} + \int \frac{|\partial_y u_n|^2}{y^4(1+y^4)(1+|\log y|)^2} \geq c$$

for some positive constant $c > 0$. Since u_n weakly converges to u_∞ in H^4_{loc} on any compact subinterval of $y \in (0, \infty)$, we can pass to the limit to conclude

$$\int \frac{|\partial_y Hu_\infty|^2}{1+y^4} + \int \frac{Hu_\infty^2}{1+y^8} + \int \frac{|u_\infty|^2}{y^2(1+y^8)(1+|\log y|)^2} + \int \frac{|\partial_y u_\infty|^2}{y^4(1+y^4)(1+|\log y|)^2} \geq c.$$

This contradicts the established identity $u_\infty = 0$ and concludes the proof of Lemma A.7.

A.5. **Coercivity of H.** We complement the coercivity property of the operator H^2, established in the previous section, by the corresponding statement for the operator H, which follows from standard compactness argument. A complete proof is given in [5] with a slightly different orthogonality condition but the proof is the same and therefore left to the reader.

Lemma A.9 (Coercivity of H)**.** *Let $M \geq 1$ be fixed. Then there exists $c(M) > 0$ such that the following holds true. Let $u \in H^2_{rad}$ with*

$$(u, \Phi_M) = 0$$

then

$$\int |\partial_{yy} u|^2 + \int \frac{|\partial_y u|^2}{y^2} + \int \frac{u^2}{y^4(1+|\log y|)^2} \leq c(M) \int |Hu|^2. \tag{A.26}$$

Appendix B. Interpolation estimates

In this appendix, we prove interpolation estimates for ε in the bootstrap regimes which are used all along the proof of Proposition 4.2. We recall the norm $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_4$, introduced in (3.13), together with their bootstrap bounds:

$$\begin{aligned}\mathcal{E}_1 &= \int |\nabla \varepsilon|^2 \leq K\delta(b^*),\\ \mathcal{E}_2 &= \int |H\varepsilon|^2 \leq Kb^2(t)|\log b(t)|^5,\\ \mathcal{E}_4 &= \int |H^2\varepsilon|^2 \leq K\frac{b^4(t)}{|\log b(t)|^2}.\end{aligned}$$

Lemma B.1 (Interpolation estimates)**.** *There hold -with constants a priori depending on M-:*

$$\int\frac{|H(\varepsilon)|^2}{y^4(1+|\log y|^2)}+\int\frac{|\partial_y H(\varepsilon)|^2}{y^2(1+|\log y|^2)}+\int\frac{|\partial_{yy}H(\varepsilon)|^2}{(1+|\log y|^2)} \tag{B.1}$$
$$+\int\frac{|\varepsilon|^2}{y^4(1+y^4)(1+|\log y|^2)}+\int\frac{|\partial_y^i\varepsilon|^2}{y^{8-2i}(1+|\log y|^2)}\lesssim\mathcal{E}_4,\ \ 1\leq i\leq 4,$$

$$\int\frac{|\varepsilon|^2}{y^4(1+|\log y|^2)}+\int\frac{|\partial_y^i\varepsilon|^2}{y^{4-2i}}\lesssim\mathcal{E}_2,\quad 1\leq i\leq 2, \tag{B.2}$$

$$\int_{y\geq 1}\frac{1+|\log y|^C}{y^{8-2i}(1+|\log y|^2)}|\partial_y^i\varepsilon|^2\lesssim b^4|\log b|^{C_1(C)},\quad 0\leq i\leq 2, \tag{B.3}$$

$$\int_{y\geq 1}\frac{1+|\log y|^C}{y^{6-2i}(1+|\log y|^2)}|\partial_y^i\varepsilon|^2\lesssim b^3|\log b|^{C_1(C)},\quad 0\leq i\leq 2, \tag{B.4}$$

$$\int_{y\geq 1}\frac{1+|\log y|^C}{y^{4-2i}(1+|\log y|^2)}|\partial_y^i\varepsilon|^2\lesssim b^2|\log b|^{C_1(C)},\quad 0\leq i\leq 1, \tag{B.5}$$

$$\left\|\varepsilon(1+|\log y|^C)\right\|^2_{L^\infty_{y\geq 1}}\lesssim b^2|\log b|^{C_1(C)}, \tag{B.6}$$

$$\|\varepsilon\|^2_{L^\infty_{y\leq 1}}+\|\partial_y\varepsilon\|^2_{L^\infty_{y\leq 1}}+\|y\partial_{yy}\varepsilon\|^2_{L^\infty_{y\leq 1}}\lesssim\frac{b^4}{|\log b|^2}, \tag{B.7}$$

$$\|y\partial_y\varepsilon\|^2_{L^\infty}\lesssim b^2|\log b|^5, \tag{B.8}$$

$$\left\|\frac{\varepsilon}{1+y}\right\|^2_{L^\infty}+\|\partial_y\varepsilon\|^2_{L^\infty}\lesssim b^3|\log b|^C, \tag{B.9}$$

$$\left\|\frac{\varepsilon}{1+y^2}\right\|^2_{L^\infty}+\left\|\frac{\partial_y\varepsilon}{1+y}\right\|^2_{L^\infty}+\|\partial_{yy}\varepsilon\|^2_{L^\infty_{y\geq 1}}\lesssim b^4|\log b|^C. \tag{B.10}$$

Proof. (B.1) and (B.2) are respectively direct consequences of the Lemma A.8 and Lemma A.9 and definition of the norms $\mathcal{E}_2$ and $\mathcal{E}_4$.
To prove (B.3), we split the integral at $y=B_0^{20}$.

$$\begin{aligned}&\int_{y\geq 1}\frac{1+|\log y|^C}{y^{8-2i}(1+|\log y|^2)}|\partial_y^i\varepsilon|^2\\
=\ &\int_{1\leq y\leq B_0^{20}}\frac{1+|\log y|^C}{y^{8-2i}(1+|\log y|^2)}|\partial_y^i\varepsilon|^2+\int_{y\geq B_0^{20}}\frac{1+|\log y|^C}{y^{8-2i}(1+|\log y|^2)}|\partial_y^i\varepsilon|^2\\
\lesssim\ &|\log b|^{C+2}\mathcal{E}_4\\
+\ &\left\|\frac{1+|\log y|^C}{y^2}\right\|_{L^\infty_{y\geq B_0^{20}}}\left(\int_{y\geq 1}\frac{|\partial_y^i\varepsilon|^2}{y^{8-2i}(1+|\log y|^2)}\right)^{\frac12}\left(\int_{y\geq 1}\frac{|\partial_y^i\varepsilon|^2}{y^{4-2i}(1+|\log y|^2)}\right)^{\frac12}.\end{aligned}$$

The bounds (B.1) and (B.2) conclude the proof. The bound (B.4) is a direct consequence of the last bound and the bootstrap bound for $\mathcal{E}_2$. Indeed:

$$\begin{aligned}&\int_{y\geq 1}\frac{1+|\log y|^C}{y^{6-2i}(1+|\log y|^2)}|\partial_y^i\varepsilon|^2\\
\lesssim\ &\left(\int_{y\geq 1}\frac{1+|\log y|^{2C}}{y^{8-2i}(1+|\log y|^2)}|\partial_y^i\varepsilon|^2\right)^{\frac12}\left(\int\frac{|\partial_y^i\varepsilon|^2}{y^{4-2i}(1+|\log y|^2)}\right)^{\frac12}\end{aligned}$$

and (B.4) follows.

The proof of (B.5) is the same as that of (B.3) using the energy $\mathcal{E}_1$ and $\mathcal{E}_2$. (B.6) comes from (A.2) and (B.5). Indeed:

$$\begin{aligned} & \left\|\varepsilon(1+|\mathrm{log} y|^C)\right\|^2_{L^\infty_{y\geq 1}} \\ \lesssim\ & \int_{y\geq 1}\frac{|\partial_y\varepsilon|^2(1+|\mathrm{log} y|^C)}{y^2}+\int_{y\geq 1}\frac{\varepsilon^2(1+|\mathrm{log} y|^C)}{y^4}+\int_{\frac12\leq y\leq 1}\frac{\varepsilon^2(1+|\mathrm{log} y|^C)}{y^2} \\ \lesssim\ & b^2|\mathrm{log} b|^C. \end{aligned}$$

Let us prove (B.7). Let $a\in[1;2]$ such that:

$$|\varepsilon(a)|\lesssim\int_1^2\frac{|\varepsilon(y)|}{y^3}\lesssim\sqrt{\mathcal{E}_4}. \tag{B.11}$$

Then using Cauchy-Schwarz

$$\forall y\in[0;1], |\varepsilon(y)|\lesssim|\varepsilon(a)|+\left|\int_a^y\frac{|\partial_y\varepsilon(y)|}{y^3}\right|\lesssim\sqrt{\mathcal{E}_4}.$$

In the same way, let $a\in[1;2]$ such that:

$$|\partial_y\varepsilon(a)|\lesssim\int_1^2\frac{|\partial_y\varepsilon(y)|}{y^3}\lesssim\sqrt{\mathcal{E}_4} \tag{B.12}$$

and

$$\forall y\in[0;1], |\partial_y\varepsilon(y)|\lesssim|\partial_y\varepsilon(a)|+\left|\int_a^y\frac{|\partial_{yy}\varepsilon(y)|}{y^3}\right|\lesssim\sqrt{\mathcal{E}_4}.$$

Finally, let $a\in[1;2]$ such that:

$$|\partial_{yy}\varepsilon(a)|\lesssim\int_1^2\frac{|\partial_{yy}\varepsilon(y)|}{y^2}\lesssim\sqrt{\mathcal{E}_4} \tag{B.13}$$

and

$$\forall y\in[0;1], |y\partial_{yy}\varepsilon(y)|\lesssim a|\partial_{yy}\varepsilon(a)|+\left|\int_a^y\left(\frac{|\partial_{yy}\varepsilon(y)|}{y^3}+\frac{|\partial_y^3\varepsilon(y)|}{y^2}\right)\right|\lesssim\sqrt{\mathcal{E}_4}.$$

The bound (B.8) is a direct consequence of Lemma A.1 and (B.2).

We now prove (B.9) using (A.2), (B.2) and (B.3):

$$\begin{aligned} & \left\|\frac{\varepsilon}{y}\right\|^2_{L^\infty_{y\geq 1}}+\|\partial_y\varepsilon\|^2_{L^\infty_{y\geq 1}}\lesssim\int_{y\geq 1}\left(\frac{\varepsilon^2}{y^6}+\frac{|\partial_y\varepsilon|^2}{y^4}+\frac{|\partial_{yy}\varepsilon|^2}{y^2}\right)+\int_{\frac12\leq y\leq 1}\left(\frac{|\varepsilon|^2}{y^4}+\frac{|\partial_y\varepsilon|^2}{y^2}\right) \\ \lesssim\ & \sum_{i=0}^2\left(\int_{y\geq 1}\frac{|\partial_y^i\varepsilon|^2}{y^{4-2i}(1+|\mathrm{log} y|^2)}\right)^{\frac12}\left(\int_{y\geq 1}\frac{|\partial_y^i\varepsilon|^2(1+|\mathrm{log} y|^2)}{y^{8-2i}}\right)^{\frac12}+\mathcal{E}_4\lesssim b^3|\mathrm{log} b|^C. \end{aligned}$$

Similarly, we prove (B.10) :

$$\begin{aligned} & \left\|\frac{\varepsilon}{y^2}\right\|^2_{L^\infty_{y\geq 1}}+\left\|\frac{\partial_y\varepsilon}{y}\right\|^2_{L^\infty_{y\geq 1}}+\|\partial_{yy}\varepsilon\|^2_{L^\infty_{y\geq 1}} \\ \lesssim\ & \int_{y\geq 1}\left(\frac{\varepsilon^2}{y^8}+\frac{|\partial_y\varepsilon|^2}{y^6}+\frac{|\partial_{yy}\varepsilon|^2}{y^4}+\frac{|\partial_y^3\varepsilon|^2}{y^2}\right)+\int_{\frac12\leq y\leq 1}\left(\frac{|\varepsilon|^2}{y^6}+\frac{|\partial_y\varepsilon|^2}{y^4}+\frac{|\partial_{yy}\varepsilon|^2}{y^2}\right) \\ \lesssim\ & b^4|\mathrm{log} b|^C. \end{aligned}$$

Appendix C. Localization of the profile

In this appendix, we are going to give the important steps of the proofs of Propositions 2.3 and 2.4.

To begin, remark that the definition (2.59) of $\tilde{\Psi}_b$ in the localization near B_1 gives two types of error. One is the result of the only localization. The other is the effect of the time derivative. Indeed, we can rewrite $\tilde{\Psi}_b$ as follows :

$$\tilde{\Psi}_b = \Psi_b^{(1)} + \tilde{R} \tag{C.1}$$

where

$$\Psi_b^{(1)} = -b^2(\tilde{T}_1 + b\tilde{T}_2) - \Delta\tilde{Q}_b + b\Lambda\tilde{Q}_b - (\tilde{Q}_b)^3 \tag{C.2}$$

and

$$\tilde{R} = b_s\left(3b^2\tilde{T}_3 + b\frac{\partial\tilde{T}_1}{\partial b} + b^2\frac{\partial\tilde{T}_2}{\partial b} + b^3\frac{\partial\tilde{T}_3}{\partial b}\right). \tag{C.3}$$

We compute the action of localization which produces an error localized in $[B_1, 2B_1]$ up to the term $(1-\chi_{B_1})\Lambda Q$:

$$\begin{aligned}\Psi_b^{(1)} &= \chi_{B_1}\Psi_b + b(1-\chi_{B_1})\Lambda Q + b\Lambda\chi_{B_1}\alpha - \alpha\Delta\chi_{B_1} - 2\partial_y\chi_{B_1}\partial_y\alpha \\ &+ (Q+\chi_{B_1}\alpha)^3 - Q^3 - \chi_{B_1}((Q+\alpha)^3 - Q^3).\end{aligned} \tag{C.4}$$

We estimate from the rough bounds of $(T_i)_{1\le i\le 3}$ and the choice of B_1:

$$\forall y \le 2B_1, \quad |\alpha(y)| \lesssim by\left(\frac{|\text{log}y|}{y} + \frac{by}{|\text{log}b|}\right) \lesssim b|\text{log}y|$$

and thus:

$$|b(1-\chi_{B_1})\Lambda Q + b\Lambda\chi_{B_1}\alpha - \alpha\Delta\chi_{B_1} - 2\partial_y\chi_{B_1}\partial_y\alpha| \lesssim \frac{b}{y^2}\mathbf{1}_{y\ge B_1} + b^2\text{log}y\mathbf{1}_{B_1\le y\le 2B_1},$$

$$\begin{aligned}\left|(Q+\chi_{B_1}\alpha)^3 - Q^3 - \chi_{B_1}((Q+\alpha)^3 - Q^3)\right| &\lesssim \frac{|\alpha(y)|}{y^3}|\mathbf{1}_{B_1\le y\le 2B_1}. \\ &\lesssim \frac{b\text{log}y}{y^2}\mathbf{1}_{B_1\le y\le 2B_1}\end{aligned}$$

Hence, $\Psi_b^{(1)}$ verifies the bounds (2.61), (2.62), (2.63). For the control of time derivatives, we have to use that :

$$\frac{\partial c_b}{\partial b} = O\left(\frac{1}{b|\text{log}b|^2}\right), \quad \frac{\partial d_b}{\partial b} = O\left(\frac{1}{b^2|\text{log}b|^2}\right), \tag{C.5}$$

which are consequences of their definition, and that :

$$\frac{\partial\Sigma_b}{\partial b} = O\left(\frac{1}{b|\text{log}b|}\mathbf{1}_{y\le\frac{B_0}{2}} + \frac{1}{y^2b^2|\text{log}b|^2}\mathbf{1}_{\frac{B_0}{2}\le y\le 6B_0}\right). \tag{C.6}$$

Using (C.5), (C.6) together the explicit formula of $(T_i)_{1\le i\le 3}$ yields (2.61), (2.62), (2.63) without difficulty. Only an estimate is more delicate, and requests more cancellation. Indeed, we must use that

$$H^2\left(\frac{\partial\tilde{T}_2}{\partial b}\right) = H\left(\frac{\partial\Sigma_2}{\partial b}\right) = H\left(\frac{\partial\Sigma_b}{\partial b}\right),$$

and similarly

$$H^2\left(\frac{\partial\tilde{T}_3}{\partial b}\right) = H\left(\frac{\partial\Sigma_3}{\partial b}\right) = \Lambda\left(\frac{\partial\Sigma_b}{\partial b}\right).$$

The proof of (2.64) is left to the reader. For the proof of Proposition 2.4, we just remark that, by definition :

$$\zeta_b = (\chi_{B_1} - \chi_{B_0})(bT_1 + b^2T_2 + b^3T_3)$$

This proof is afterwards the same as the previous one.

References

[1] Van den Berg, G.J.B.; Hulshof, J.; King, J., Formal asymptotics of bubbling in the harmonic map heat flow, SIAM J. Appl. Math. vol 63, o5. pp 1682-1717.,

[2] Brezis, H., Analyse fonctionelle, Masson (1983).

[3] Cote, R.; Martel, Y.; Merle, F., Construction of multisolitons solutions for the L^2-supercritical gKdV and NLS equations, arXiv:0910.2594 (2009)

[4] Herrero, M.A.; Velázquez, J.J.L.: Explosion de solutions des équations paraboliques semilinéaires supercritiques. C. R. Acad. Sci. Paris 319, 141–145 (1994).

[5] Hillaret, M.; Raphaël, P., Smooth type II blow-up solutions to the four dimensional energy Phys. (2010), 300, no 1, 205-242.

[6] Krieger, J.; Schlag, W.; Tataru, D. Renormalization and blow-up for charge one equivariant critical wave maps, Invent. Math. 171 (2008), no. 3, 543–615

[7] Krieger, J.; Schlag, W., On the focusing critical semi-linear wave equation, Amer. J. Math. 129 (2007), no. 3, 843–913.

[8] Matano, H.; Merle, F., Classification of type I and type II behaviors for a supercritical nonlinear heat equation. J. Funct. Anal. 256 (2009), no. 4, 992–1064.

[9] Matano, H.; Merle, F., On nonexistence of type II blowup for a supercritical nonlinear heat equation. Comm. Pure Appl. Math. 57 (2004), no. 11, 1494–1541.

[10] Mizoguchi, N., Rate of type II blowup for a semilinear heat equation, Math. Ann. 339 (2007), no. 4, 839–877.

[11] Merle, F.; Raphaël, P., blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation, Ann. Math. 161 (2005), no. 1, 157-222.

[12] Merle, F.; Raphaël, P., Sharp upper bound on the blow-up rate for critical nonlinear Schrödinger equation, Geom. Funct. Anal. 13 (2003), 591-642.

[13] Merle, F.; Raphaël, P., On universality of blow-up profile for L^2 critical nonlinear Schrödinger equation, Invent. Math. 156, 565-672 (2004).

[14] Merle, F.; Raphaël, P., Sharp lower bound on the blow-up rate for critical nonlinear Schrödinger equation, J. Amer. Math. Soc. 19 (2006), no. 1, 37-90.

[15] Merle, F.; Raphaël, P., Profiles and quantization of the blow-up mass for critical nonlinear Schrödinger equation, Comm. Math. Phys. 253 (2005), no. 3, 675-704.

[16] Merle, F.; Raphaël, P.; Rodnianski, I., blow-up dynamics for smooth solutions to the energy critical Schrödinger map, preprint 2011.

[17] Perelman, G., On the blow-up phenomenon for the critical nonlinear Schrödinger equation in 1D, Ann. Henri. PoincarÈ, 2 (2001), 605-673.

[18] Raphaël, P.; Rodnianski, I., Stable blow-up dynamics for the critical corotational wave maps and equivariant Yang Mills problems, to appear in Prep. Math. IHES.

[19] Raphaël, P.; Schweyer, R., Stable blow-up dynamics for the 1-corotational heat flow, preprint 2011.

[20] Rodnianski, I., Sterbenz, J., On the formation of singularities in the critical $O(3)$ σ-model, Ann. of Math. (2) 172 (2010), no. 1, 187–242.

Institut de Mathématiques de Toulouse, Université Paul Sabatier, Toulouse, France

E-mail address: `remi.schweyer@math.univ-toulouse.fr`