

STABLE BLOW-UP DYNAMIC FOR THE PARABOLIC-PARABOLIC PATLAK-KELLER-SEGEL MODEL

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ABSTRACT. We consider the parabolic-parabolic two-dimensional Patlak-Keller-Segel problem. We prove the existence of stable blow-up dynamics in finite time in the radial case. We extend in this article the result of [36] for the parabolic-elliptic case.

1. Introduction

1.1. Setting of the problem. The Patlak-Keller-Segel model, which is studied in this article, is one of the simplest modelisation of chemotaxis, suggested by Patlak in 1953 [33] and by Keller and Segel in [24], [25] and [26]. More precisely, the cells of living organisms communicate to each other through chemical species. When these chemical species cause a collective movement of the cells, chemotaxis is the term applied to describe this phenomenon. Chemotaxis plays a crucial role in a large number of biological situations, like angiogenesis, embryonic development or formation of colonies of bacteria. The interested reader can refer to [34], [41], [42] and [19] for more detailed information.

In this paper, we study a particular case of chemotaxis, in which the chemical species is directly product by the cells. This implies a strong coupling between the spatio-temporal dynamics of cells and the chemoattractant. The most famous example is the amoeba *Dictyostelium discoideum*, which has attracted a considerable attention for the past forty years. A complete review is available in [20] and [21]. The following model is suggested by Nanjundiah in [32] :

$$\begin{cases} \partial_t u = \nabla \cdot (\kappa \nabla u + \chi u \nabla v), \\ \partial_t v = \eta \Delta v - \beta u + \alpha v, \\ u(0, x) = u_0 > 0, \\ v(0, x) = v_0 > 0, \end{cases} \quad (t, x) \in \mathbb{R}^+ \times \Omega. \quad (1.1)$$

where Ω is an open set of \mathbb{R}^2 or the whole plane, u is the density of amoeba, v the concentration of the chemoattractant, κ and η are respectively diffusion coefficients for the amoeba and the chemoattractant, χ the sensitivity of the amoeba to the chemoattractant, β the rate of production of chemoattractant per amoeba and α the rate of destruction of the chemoattractant. All above quantities are positive. For biological reasons, the last rate α is very small, and in a first approximation, we can consider $\alpha = 0$. Using a suitable rescaling and an adimensionalization, we obtain:

$$\begin{cases} \partial_t u = \nabla \cdot (\nabla u + u \nabla v), \\ c \partial_t v = \Delta v - u, \\ u(0, x) = u_0 > 0, \\ v(0, x) = v_0 > 0, \end{cases} \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2. \quad (1.2)$$

where $c = \frac{\kappa}{\eta}$ is the difference of the time scales of the diffusive processes undergone by u and v .

1.2. On the parabolic-elliptic model. In this subsection we take $c = 0$, and the system (1.2) becomes:

$$(PKSpe) \quad \begin{cases} \partial_t u = \nabla \cdot (\nabla u + u \nabla v), \\ v = \frac{1}{2\pi} \log|x| \star u \\ u|_{t=0} = u_0 > 0 \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \quad (1.3)$$

In a pioneering work [22] is proved for the first time that the corresponding solution to (1.3) with small enough mass is global in time, and blow-up can occur for initial data with large mass. A important step was taken in [2] and [11] with the establishment of the logarithmic Hardy-Littlewood-Sobolev inequality: $\forall u \geq 0$ with $\int u = M$,

$$\int u \log u + \frac{4\pi}{M} \int \phi_u u \geq M [1 + \log \pi - \log M] \quad (1.4)$$

where given u we defined:

$$\phi_u = \frac{1}{2\pi} \log|x| \star u.$$

The left term coincides with the free energy for $M = 8\pi$, and thus, the free energy is lower bounded for this mass. Furthermore Q is up to symmetry the unique minimizer, where Q is a radial explicit profile defined by:

$$Q(x) = \frac{8}{(1 + |x|^2)^2}. \quad (1.5)$$

This is the key ingredient in the proof that all solutions of (1.3) such that $M \leq 8\pi$ are global in time. [15], [16] and [8]. Moreover, for $M < 8\pi$, zero is a local universal attractor. The dynamics of these solutions is sharply described in [10].

Now, the solutions with enough good decay, *ie* solutions with finite second moment $\int_{\mathbb{R}^2} |x|^2 u_0(x) dx < \infty$, satisfy the virial law:

$$\frac{d}{dt} \int |x|^2 u(t, x) dx = 4 \left(1 - \frac{M}{8\pi}\right) M.$$

This argument gives two informations. First, if the mass of the solution is 8π and the second moment is finite, then, this second moment is preserved. Secondly, if the mass is larger than 8π and the second moment is finite, the solution blows-up in finite time.

In the case of $M = 8\pi$, there are two conservation laws : the mass and the second moment. The problem is both L^1 critical, and energy critical. In [7], the authors prove in the case of finite second moment, the solutions grow-up and converge to a Dirac mass at infinity time. The argument is not constructive and gives no information about the rate of convergence. The article [40] answers this question. In the case of bounded domain, the situation is described in [5] and [23].

In the case of infinite second moment is proved in [6] the existence of global solutions converging to the ground state, and the rate of convergence is proved in [12]. The case of very slow decay with finite mass 8π is an open question.

For the solution with large mass, *ie* $M > 8\pi$, we have seen that if the second moment is finite, the solution blows-up in finite time. In fact, this argument can be extended in the case of all solutions with large mass. The problem is still poorly understood. In the radial case, a first example of blow-up solution in finite time is described in [17] using formal matching asymptotic, and completed in [43] and [44]. Recently, in [36] is given a sharp description of the blow-up dynamics, and in particular its stability.

1.3. On the parabolic-parabolic model. In [30] is proved by fixed point arguments the existence and the uniqueness of local non negative smooth solutions of (1.2). Moreover, if we consider initial data with enough fast decay, *ie* $u_0 \in L^1(\mathbb{R}^2)$, there is the conservation of the mass :

$$\int_{\mathbb{R}^2} u(t, x) dx = \int_{\mathbb{R}^2} u(0, x) dx = M \quad (1.6)$$

Now, if (u, v) is solution to the problem (1.2), then the rescaled solution

$$\begin{cases} u_\lambda \\ v_\lambda \end{cases} = \begin{cases} \lambda^2 u(\lambda^2 t, \lambda x) \\ v(\lambda^2 t, \lambda x) \end{cases}$$

is also a solution. Furthermore, the L^1 norm is unchanged by this scaling :

$$\int_{\mathbb{R}^2} u_\lambda(t, x) dx = \int_{\mathbb{R}^2} u(\lambda^2 t, x) dx$$

The problem is thus L^1 critical. The second important quantity is the free energy functional :

$$E(u, v) = \int_{\mathbb{R}^2} u(x) \log u(x) dx + \int_{\mathbb{R}^2} u(x) v(x) dx - \frac{1}{2} \int_{\mathbb{R}^2} v(x) \Delta v(x) dx$$

This functional play a crucial role in nonlinear diffusion and kinetic models [1]. This energy is dissipated by the flow. Moreover, the problem is almost energy critical in the following sense:

$$E(u_\lambda, v_\lambda) = E(u, v) + M \left(2 - \frac{M}{4\pi} \right) \log \lambda.$$

For initial data with small mass $M < 8\pi$, corresponding solutions of (1.2) exist globally in time [9]. See [29] and [3] for similar results on a disk.

Now, the threshold effect of the mass on the dynamics of the solutions of (1.2) is more imprecise than the parabolic-elliptic case. Indeed, self-similar solutions with $M > 8\pi$ are exhibited ([31] in the case $c = 1$, [4] for all $c > 0$). Hence, in opposite of the parabolic-elliptic case, solution with large mass can exist globally in time.

A blow-up dynamic in finite time is proved in a bounded domain in [18], using formal matching asymptotic as [17] for the parabolic-elliptic case.

1.4. Main result. Our result concerns the parabolic-parabolic model, in the case $c = 1$, *ie* :

$$\begin{cases} \partial_t u = \nabla \cdot (\nabla u + u \nabla v), \\ \partial_t v = \Delta v - u, \\ u(0, x) = u_0 > 0, \\ v(0, x) = v_0 > 0, \end{cases} \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2. \quad (1.7)$$

In the continuation of [36] for the parabolic-elliptic case, we obtain a sharp description of the blow-up dynamics in finite time, for data with small super critical mass. In particular, we obtain the rate of convergence and the stability of the blow-up, for small perturbation in the energy space. We use a similar approach like in [39] for the energy critical semilinear heat flow, and in [37] and [38] for the harmonic heat flow.

Let the weighted H^2 space:

$$\|\varepsilon\|_{H_Q^2} = \|(1 + r^2) \Delta \varepsilon\|_{L^2} + \|(1 + r) \nabla \varepsilon\|_{L^2} + \|\varepsilon\|_{L^2} \quad (1.8)$$

and the weighted H^3 space:

$$\|\eta\|_N = \|(1 + r) \nabla \Delta \eta\|_{L^2} + \|\Delta \eta\|_{L^2}. \quad (1.9)$$

We introduce the energy norm

$$\|\boldsymbol{\varepsilon}\|_{\mathcal{E}} = \|\varepsilon\|_{H_Q^2} + \|\eta\|_N + \|\varepsilon\|_{L^1} = \|(\varepsilon, \eta)\|_{W_Q} + \|\varepsilon\|_{L^1}. \quad (1.10)$$

Theorem 1.1 (Stable chemotactic blow up). *There exists a set of initial data of the form*

$$\mathbf{u}_0 = \begin{vmatrix} u_0 \\ v_0 \end{vmatrix} = \mathbf{Q} + \boldsymbol{\varepsilon}_0, \quad \boldsymbol{\varepsilon}_0 = \begin{vmatrix} \varepsilon_0 \\ \eta_0 \end{vmatrix} \in \mathcal{E}, \quad u_0 > 0, \quad v_0 > 0, \quad \|\boldsymbol{\varepsilon}_0\|_{\mathcal{E}} \ll 1$$

such that the corresponding solution $u \in \mathcal{C}([0, T], \mathcal{E})$ to (1.2) with $c = 1$ satisfies the following:

(i) Small super critical mass:

$$8\pi < \int u_0 < 8\pi + \alpha^*$$

for some $0 < \alpha^* \ll 1$ which can be chosen arbitrarily small;

(ii) Blow up : the solution blows up in finite time $0 < T < +\infty$;

(iii) Universality of the blow up bubble: the solution admits for all times $t \in [0, T)$ a decomposition

$$\mathbf{u}(t, x) = \begin{vmatrix} \frac{1}{\lambda^2(t)}(Q + \varepsilon) \left(t, \frac{x}{\lambda(t)} \right) \\ (\phi_Q + \eta) \left(t, \frac{x}{\lambda(t)} \right) \end{vmatrix} \quad (1.11)$$

with

$$\|(\varepsilon(t), \eta(t))\|_{W_Q} \rightarrow 0 \quad \text{as } t \rightarrow T \quad (1.12)$$

and the universal blow up speed:

$$\lambda(t) = \sqrt{T - t} e^{-\sqrt{\frac{|\log(T-t)|}{2}} + O(1)} \quad \text{as } t \rightarrow T. \quad (1.13)$$

(iv) Stability: the above blow up dynamics is stable by small perturbation of the data in \mathcal{E} :

$$\mathbf{f}_0 = \begin{vmatrix} f_0 \\ g_0 \end{vmatrix} \quad f_0 > 0, \quad g_0 > 0 \quad \|\mathbf{f}_0 - \mathbf{u}_0\|_{\mathcal{E}} < \epsilon(\mathbf{u}_0).$$

Comments on the result

- (1) To our knowledge, this is the first proof of the existence of blow-up dynamics in finite time for the parabolic-parabolic Patlak-Keller-Segel in the whole plane. Very recently, Mizogushi and Winkler have obtained the existence of blow-up solution with a virial argument, in the case where α , the rate of destruction of the chemoattractant, is non negative. Moreover this kind of obstructive argument does not come with a sharp description of the blow up bubble like (1.13).
- (2) In this work, for the sake of simplicity, we have chosen $c = 1$, where we recall that c is the difference of the time scales of the diffusive processes undergone by u and v . We have seen the crucial role of this constant in [4] for the existence of self-similar solutions. In fact, the theorem is true for all $c > 0$. More precisely, we observe that this quantity doesn't influence on the leading order of the dynamics of the solution. To understand this result, a possible line of reasoning is that, if we fix the constant c , we can find initial data satisfying the conditions of the theorem enough concentrated such that the propagation speed of the chemoattractant becomes negligible.

- (3) Comparing Theorem 1.1 and the blow up result in [36], we see that the parabolic and elliptic couplings yield *in the regime we consider* the same kind of blow up bubble to leading order. Let us stress that this has no reason to hold in general. A celebrated example is the case of the Zakahrov equations of plasma physics:

$$\begin{cases} i\partial_t u + \Delta u - nu = 0 \\ \frac{1}{c_0^2} \partial_{tt} n - \Delta n = \Delta |u|^2 \end{cases}, \quad x \in \mathbb{R}^2$$

which in the limit $c_0 \rightarrow +\infty$ reduce to the mass critical nonlinear Schrödinger equation. Merle proved in [28] that for all $c_0 > +\infty$ (wave coupling), the stable log log regime of the $c_0 = +\infty$ case (elliptic coupling) is destroyed, hence showing the importance of the nature of the coupling.

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1.5. Notations. In this problem, we study a couple of solution (u, v) . We notice without any difference

$$(u, v) = \begin{vmatrix} u \\ v \end{vmatrix} = \mathbf{u},$$

where the last notation will be used only in the case where there is no possible confusion. In the same way, we notice for an operator F :

$$F(u, v) = \begin{vmatrix} F^{(1)}(u, v) \\ F^{(2)}(u, v) \end{vmatrix}$$

We use the real $L^2 \times \dot{H}^1$ scalar product :

$$\left\langle \begin{vmatrix} u \\ v \end{vmatrix}, \begin{vmatrix} f \\ g \end{vmatrix} \right\rangle = \int u f + \int \nabla v \nabla g = \int_0^\infty u(r) f(r) r dr + \int_0^\infty \nabla v(r) \nabla g(r) r dr.$$

For a given function u , we note its Poisson field

$$\phi_u = \frac{1}{2\pi} \log|x| \star u.$$

We let $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ be a radially symmetric cut off function with

$$\chi(r) = \begin{cases} 1 & \text{for } r \leq 1, \\ 0 & \text{for } r \geq 2 \end{cases}, \quad \chi(r) \geq 0.$$

Moreover, for a given $B > 0$, we let

$$\chi_B(r) = \chi\left(\frac{r}{B}\right)$$

Given $b > 0$, we let

$$B_0 = \frac{1}{\sqrt{b}}, \quad B_1 = \frac{|\log b|}{\sqrt{b}}. \quad (1.14)$$

Finally, we use the scaling operator:

$$\Lambda \begin{vmatrix} f \\ g \end{vmatrix} = \left(\begin{vmatrix} \frac{df_\lambda}{d\lambda} \\ \frac{dg_\lambda}{d\lambda} \end{vmatrix} \right)_{\lambda=1} = \begin{vmatrix} 2f + y \cdot \nabla f \\ y \cdot \nabla g \end{vmatrix} = \begin{vmatrix} \nabla \cdot (yf) \\ y \cdot \nabla g \end{vmatrix} \quad (1.15)$$

The article is organized as follow. In section 2, we obtain spectral gap estimates for linearized operators close to the ground state. In section 3, we construt an enough good approximate profiles four our analysis. In section 4, we introduce the energy method based on a bootstrap argument. A keystone of the proof is a monotonicity formula, which we prove in section 5. We conclude the bootstrap argument in section 6, and concludes the proof of the Theorem 1.1 in section 7.

2. Spectral gap estimates

In this section, we obtain spectral gap estimates for linearized operators, which are a keystone of the energy method to control the gap between the solution and the approximate profile, which shall construct in the next section.

2.1. On the linearized energy. In this subsection, we start with the introduction of the operator \mathcal{M} coming from the linearization of free energy around the ground state on the suitable space. Then, the structure of this operator is going to be studied, before giving a proposition on its subcoercivity.

Let the suitable space X_Q defined by:

$$X_Q = \left\{ (u, v) \in L_Q^2 \times H^1, \text{ s.t. } \int u = 0, \text{ and } \int (1 + |\log r|)^2 |\nabla v|^2 < \infty \right\}.$$

We introduce the suitable norm:

$$\|(u, v)\|_{X_Q} = \|u\|_{L_Q^2} + \|v\|_{\dot{H}^1}. \quad (2.1)$$

From the Lemma B.1, we know that $\forall (u, v) \in X_Q, \nabla \phi_u \in L^2$. Let's prove that the ground state is a local minima of the free energy for deformations in the space X_Q . Let $(u, v) \in X_Q$, and let the function F define for small $\lambda \in \mathbb{R}$ such that $Q + \lambda u > 0$ by:

$$\begin{aligned} F(\lambda) &= E(Q + \lambda u, \phi_Q + \lambda v) \\ &= \int (Q + \lambda u) \log(Q + \lambda u) + \int (Q + \lambda u)(\phi_Q + \lambda v) - \frac{1}{2} \int (\phi_Q + \lambda v) \Delta(\phi_Q + \lambda v) \end{aligned}$$

We compute

$$\begin{aligned} F'(\lambda) &= \int u [\log(Q + \lambda u) + 1] + \int u(\phi_Q + \lambda v) + \int (Q + \lambda u)v \\ &\quad - \frac{1}{2} \int v \Delta(\phi_Q + \lambda v) - \frac{1}{2} \int (\phi_Q + \lambda v) \Delta v \\ &= \int u [\log(Q + \lambda u) - \log 8 + \phi_Q] + \frac{1}{2} \int (vQ - \Delta v \phi_Q) + \lambda \int \{2uv - v \Delta v\} \end{aligned}$$

Hence, using the explicit expression of Q and ϕ_Q , we obtain

$$F'(0) = \frac{1}{2} \int (vQ - \Delta v \phi_Q).$$

We shall prove that the integration by part is valid for $(u, v) \in X_Q$. In this purpose, let $R_0 > 0$. We compute

$$\int_{R_0}^{R_0+1} \left\{ \int_0^R \Delta v \phi_Q \right\} dR = \int_{R_0}^{R_0+1} \left\{ \int_0^R vQ \right\} dR + \int_{R_0}^{R_0+1} [\phi_Q r v'(r) - \nabla \phi_Q v(r) r]_0^R dR.$$

Now, as $(u, v) \in X_Q$, $v \in L^2$ and $\int (1 + |\log r|)^2 |\nabla v|^2 < \infty$ yield

$$\lim_{R_0 \rightarrow +\infty} \int_{R_0}^{R_0+1} [\phi_Q r v'(r) - \nabla \phi_Q v(r) r]_0^R dR = 0.$$

Thus : $\int v Q = \int \Delta v \phi_Q$, and thus Q is a critical point:

$$F'(0) = 0.$$

We can compute the Hessian, using that $(\nabla \phi_u, \nabla v) \in L^2 \times L^2$:

$$\begin{aligned} F''(\lambda)|_{\lambda=0} &= \int \frac{u^2}{Q} + 2uv - \int v \Delta v \\ &= \int \left(\frac{u^2}{Q} + u \phi_u + u(v - \phi_u) + uv - v \Delta v \right) \\ &= \int \left(\frac{u^2}{Q} + u \phi_u \right) + \int |\nabla v - \nabla \phi_u|^2 \\ &= \int \left(\frac{u^2}{Q} + vu \right) + \int \nabla(v - \phi_u) \nabla v = \left\langle \mathcal{M} \begin{vmatrix} u \\ v \end{vmatrix}, \begin{vmatrix} u \\ v \end{vmatrix} \right\rangle. \end{aligned}$$

In [36] is proved that for a function u such that $\int u = 0$, $\int \left(\frac{u^2}{Q} + u \phi_u \right) \geq 0$. Then, from the third line of the above compute, $F''(\lambda)|_{\lambda=0} \geq 0$ and thus (Q, ϕ_Q) is a local minima of the free energy for deformations in the space X_Q . Furthermore, the operator \mathcal{M} define by

$$\mathcal{M} : \begin{vmatrix} \frac{u}{Q} + v \\ v - \phi_u \end{vmatrix} = \begin{vmatrix} \mathcal{M}^{(1)}(u, v) \\ \mathcal{M}^{(2)}(u, v) \end{vmatrix} \quad (2.2)$$

is a positive operator on X_Q . The following lemma describes the structure of this operator.

Lemma 2.1 (Structure of the linearized energy). *Let the operator*

$$\mathcal{M} : \begin{vmatrix} \frac{u}{Q} + v \\ v - \phi_u \end{vmatrix} \quad (2.3)$$

Then:

$$\int Q |\mathcal{M}^{(1)}(u, v)|^2 + \int |\nabla \mathcal{M}^{(2)}(u, v)|^2 \lesssim \|(u, v)\|_{X_Q}^2 + \int \frac{v^2}{1+y^4}, \quad (2.4)$$

Moreover, there holds:

(i) *Self-adjointness:*

$$\forall (\mathbf{u}, \mathbf{v}) \in L_Q^2 \times H^1, \quad \langle \mathcal{M} \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathcal{M} \mathbf{v} \rangle. \quad (2.5)$$

(ii) *Algebraic identities:*

$$\mathcal{M}(\Lambda Q) = \mathcal{M} \begin{vmatrix} \Lambda Q \\ \phi_{\Lambda Q} \end{vmatrix} = \begin{vmatrix} -2 \\ 0 \end{vmatrix} \quad (2.6)$$

(iii) *Generalized kernel: Let $(u, v) \in X_Q$ such that :*

$$\begin{cases} \nabla \left(\frac{u}{Q} + v \right) = 0 \\ \Delta v = u \end{cases}$$

Then

$$\mathbf{u} \in \text{Span}(\Lambda Q) \quad (2.7)$$

Proof. Let $(u, v) \in X_Q$. The continuity of this operator follows from the Lemma B.1:

$$\begin{aligned} |\langle \mathcal{M}\mathbf{u}, \mathcal{M}\mathbf{u} \rangle| &= \left| \int Q \left(\frac{u}{Q} + v \right)^2 + \int (\nabla v - \nabla \phi_u)^2 \right| \\ &\lesssim \int \frac{u^2}{Q} + \int Qv^2 + \int |\nabla v|^2 + \int |\nabla \phi_u|^2 \\ &\lesssim \int \frac{u^2}{Q} + \int |\nabla v|^2 + \int \frac{v^2}{1+y^4}. \end{aligned}$$

To prove the self-adjointness of the operator \mathcal{M} , let $\{(u, v); (f, g)\} \in (L_Q^2 \times H^1)^2$, and we compute:

$$\begin{aligned} \left\langle \mathcal{M} \begin{vmatrix} u \\ v \end{vmatrix}, \begin{vmatrix} f \\ g \end{vmatrix} \right\rangle &= \int \left(\frac{u}{Q} + v \right) f + \int |\nabla v - \nabla \phi_u| \nabla g \\ &= \int \frac{uf}{Q} + \int \nabla v \nabla g + \int vf + \int gu. \\ &= \left\langle \begin{vmatrix} u \\ v \end{vmatrix}, \mathcal{M} \begin{vmatrix} f \\ g \end{vmatrix} \right\rangle \end{aligned}$$

The last equality comes from $\int f = \int u = 0$. (2.6) is a consequence of the explicit expression for ΛQ and $\phi_{\Lambda Q}$.

Let's prove (2.7). For this purpose, let $(u, v) \in X_Q$ such that:

$$\begin{cases} \nabla \left(\frac{u}{Q} + v \right) = 0 \\ \Delta v = u \end{cases}$$

From the Lemma B.2, $(u, v) \in X_Q$ yields $\phi_{\Delta v} = v$ and thus

$$\phi_u = v. \tag{2.8}$$

Prove (2.7) is equivalent to find the functions $u \in L_Q^2$ such that :

$$\nabla \left(\frac{u}{Q} + \phi_u \right) = 0. \tag{2.9}$$

In the framework of radial functions, let the partial mass be :

$$m_u(r) = \int_0^r u(\tau) \tau d\tau. \tag{2.10}$$

Remark that $\phi'_u(r) = \frac{m_u(r)}{r}$ and $u(r) = \frac{m'_u(r)}{r}$. Hence,

$$\left(\frac{u}{Q} + \phi_u \right)' = \frac{u'(r)}{Q} - \frac{Q'}{Q^2} u(r) + \phi'_u(r) = \frac{1}{Q} \left(\frac{m_u(r)}{r} \right)' - \frac{Q'}{Q^2} \frac{m'_u(r)}{r} + \frac{m_u(r)}{r} = 0$$

Thus, the equation (2.9) becomes:

$$L_0 m_u = 0$$

where

$$\begin{aligned} L_0 m_u &= -m''_u + \left(\frac{1}{r} + \frac{Q'}{Q} \right) m'_u - Q m_u \\ &= -m''_u - \frac{3r^2 - 1}{r(1+r^2)} m'_u - \frac{8}{(1+r^2)^2}. \end{aligned} \tag{2.11}$$

The basis of solutions to this homogeneous equation is explicit and given by:

$$\psi_0(r) = \frac{r^2}{(1+r^2)^2}, \quad \psi_1(r) = \frac{1}{(1+r^2)^2} [r^4 + 4r^2 \log r - 1]. \quad (2.12)$$

The regularity of u at the origin, and thus the regularity of m_u implies that $m_u \in \text{Span}(\psi_0)$. To remark that $m_{\Lambda Q} = 8\psi_0$ conclude the proof of (2.7) and hence the proof of the Lemma 2.1. \square

Before studying the linearized operator \mathcal{L} close to the ground state of the (PKS) flow, we prove sub-coercivity for the operator \mathcal{M} which are the key to the proof of coercive estimates for these operators under additional orthogonality conditions.

Proposition 2.2 (Sub-coercivity of the operator \mathcal{M}). *There exists a universal constant $\delta_0 > 0$ such that for all $\mathbf{u} \in X_Q$,*

$$\langle \mathcal{M}\mathbf{u}, \mathbf{u} \rangle \geq \delta_0 \left(\int \frac{u^2}{Q} + \int |\nabla v|^2 \right) - \frac{1}{\delta_0} \langle \mathbf{u}, \Lambda \mathbf{Q} \rangle^2. \quad (2.13)$$

Proof. Step 1 : Coercivity

To begin, prove that :

$$I = \inf \{ \langle \mathcal{M}\mathbf{u}, \mathbf{u} \rangle, \quad \|\mathbf{u}\|_{X_Q} = 1, \quad \langle \mathbf{u}, \Lambda \mathbf{Q} \rangle = 0 \} > 0. \quad (2.14)$$

We argue by contradiction. Let a sequence $\begin{cases} u_n \\ v_n \end{cases} = \mathbf{u}_n \in X_Q$ such that:

$$0 \leq \langle \mathcal{M}\mathbf{u}_n, \mathbf{u}_n \rangle \leq \frac{1}{n}, \quad \|\mathbf{u}_n\|_{X_Q} = 1, \quad \langle \mathbf{u}_n, \Lambda \mathbf{Q} \rangle = 0.$$

We recall that:

$$\langle \mathcal{M}\mathbf{u}_n, \mathbf{u}_n \rangle = (\mathcal{M}_0 u_n, u_n) + \int |\nabla v_n - \nabla \phi_{u_n}|^2.$$

Up to a subsequence,

$$\mathbf{u}_n \rightharpoonup \mathbf{u} = \begin{cases} u \\ v \end{cases} \quad \text{in } X_Q \quad (2.15)$$

Hence,

$$\int \frac{u^2}{Q} + \int |\nabla v|^2 \leq \liminf_{n \rightarrow +\infty} \int \frac{u_n^2}{Q} + \int |\nabla v_n|^2 = 1 \quad \text{and} \quad \langle \mathbf{u}, \Lambda \mathbf{Q} \rangle = 0 \quad (2.16)$$

From standard argument (see for example the proof of the Proposition 2.3 of [36]) :

$$\nabla \phi_{u_n} \rightarrow \nabla \phi_u \quad \text{in } L^2.$$

As

$$\nabla v_n \rightharpoonup \nabla v \quad \text{in } L^2,$$

we obtain :

$$\int u_n v_n = - \int \nabla \phi_{u_n} \nabla v_n \rightarrow - \int \nabla \phi_u \nabla v = \int uv \quad (2.17)$$

The positivity of the operator \mathcal{M} together with (2.16) yields

$$\langle \mathcal{M}\mathbf{u}, \mathbf{u} \rangle = 0, \quad \int \frac{u^2}{Q} + \int |\nabla v|^2 = -2 \int uv \leq 1. \quad (2.18)$$

From the normalization of the sequence we have:

$$-2 \int u_n v_n = \int \frac{u_n^2}{Q} + \int |\nabla v_n|^2 - \langle \mathcal{M}\mathbf{u}_n, \mathbf{u}_n \rangle \geq 1 - \frac{1}{n}. \quad (2.19)$$

Thus, the function \mathbf{u} verifies:

$$\langle \mathcal{M}\mathbf{u}, \mathbf{u} \rangle = 0, \quad \|\mathbf{u}\|_{X_Q} = 1 \text{ and } \langle \mathbf{u}, \Lambda\mathbf{Q} \rangle = 0 \quad (2.20)$$

From the Lemma 2.1, the condition $\langle \mathcal{M}\mathbf{u}, \mathbf{u} \rangle = 0$ yields that $\mathbf{u} = c\Lambda\mathbf{Q}$, and the orthogonality condition $\langle \mathbf{u}, \Lambda\mathbf{Q} \rangle = 0$ impose that $c = 0$. Hence, $u = 0$ which contradicts $\|\mathbf{u}\|_{X_Q} = 1$. This concludes the proof of (2.14).

Step 2 : Conclusion

Consider $\mathbf{u} \in X_Q$. Let

$$\mathbf{v} = \mathbf{u} - \frac{\langle \mathbf{u}, \Lambda\mathbf{Q} \rangle}{\langle \Lambda\mathbf{Q}, \Lambda\mathbf{Q} \rangle} \Lambda\mathbf{Q} \quad (2.21)$$

By construction, $\langle \mathbf{v}, \Lambda\mathbf{Q} \rangle = 0$ and thus from (2.14)

$$\langle \mathcal{M}\mathbf{v}, \mathbf{v} \rangle \geq \delta_0 \|\mathbf{v}\|_{X_Q}^2 \geq \delta_0 \|\mathbf{u}\|_{X_Q}^2 - \frac{1}{\delta_0} \langle \mathbf{u}, \Lambda\mathbf{Q} \rangle.$$

This concludes the proof of the Proposition 2.2. \square

2.2. On the linearized operator \mathcal{L} . In this section, we begin to the study of the structure of the linearized operator close to Q of the (PKS) flow for perturbations in the energy space \mathcal{E} . Moreover the operator \mathcal{L} is given by:

$$\mathcal{L}(\varepsilon, \eta) = \left| \begin{array}{c} \nabla \cdot \{Q\nabla \mathcal{M}^{(1)}(\varepsilon, \eta)\} \\ \Delta \mathcal{M}^{(2)}(\varepsilon, \eta) \end{array} \right| = \left| \begin{array}{c} \nabla \cdot \{Q\nabla \left(\frac{\varepsilon}{Q} + \eta\right)\} \\ \Delta \eta - \varepsilon \end{array} \right|$$

We can formally define its adjoint for the $L^2 \times \dot{H}^1$ scalar product by:

$$\mathcal{L}^*(\varepsilon, \eta) = \mathcal{M} \left| \begin{array}{c} \nabla \cdot \{Q\nabla \varepsilon\} \\ \Delta \eta \end{array} \right| = \left| \begin{array}{c} \frac{\nabla \cdot \{Q\nabla \varepsilon\}}{Q} + \Delta \eta \\ \Delta \eta - \phi_{\nabla \cdot \{Q\nabla \varepsilon\}} \end{array} \right| \quad (2.22)$$

Lemma 2.3 (Structure of the operator \mathcal{L}). (i) *Continuity of \mathcal{L} on \mathcal{E} :*

$$\|\mathcal{L}\varepsilon\|_{X_Q} \lesssim \|\varepsilon\|_{\mathcal{E}}. \quad (2.23)$$

(ii) *Adjunction:* $\forall(\varepsilon, \tilde{\varepsilon}) \in \mathcal{E}^2$,

$$\langle \mathcal{L}\varepsilon, \tilde{\varepsilon} \rangle = \langle \varepsilon, \mathcal{L}^*\tilde{\varepsilon} \rangle. \quad (2.24)$$

(iii) *Algebraic identities:*

$$\mathcal{L}(\Lambda\mathbf{Q}) = 0, \quad (2.25)$$

$$\forall c \in \mathbb{R}, \quad \mathcal{L}^*(1, cr^2) = 0, \quad \mathcal{L}^*\left(r^2, -4 \int_0^r \frac{\log(1 + \tau^2) d\tau}{\tau}\right) = \left| \begin{array}{c} -4 \\ 0 \end{array} \right|. \quad (2.26)$$

(iv) *Vanishing average:* $\forall \varepsilon \in \mathcal{E}$,

$$\left\langle \mathcal{L}\varepsilon, \left| \begin{array}{c} 1 \\ c \end{array} \right. \right\rangle = 0, \quad \forall c \in \mathbb{R} \quad (2.27)$$

Proof. Step 1 : Continuity :

First, we rewrite the operator \mathcal{L} with an explicit formula:

$$\mathcal{L}(\varepsilon, \eta) = \left| \begin{array}{c} \Delta \varepsilon + \varepsilon Q + \nabla \varepsilon \nabla \phi_Q + Q \Delta \eta + \nabla Q \nabla \eta \\ \Delta \eta - \varepsilon \end{array} \right|.$$

Hence,

$$\begin{aligned}
& \|\mathcal{L}(\varepsilon, \eta)\|_{X_Q}^2 \\
& \lesssim \int \frac{|\Delta\varepsilon|^2}{Q} + \int Q|\varepsilon|^2 + \int \frac{|\nabla\phi_Q|^2}{Q}|\nabla\varepsilon|^2 + \int \frac{|\nabla Q|^2}{Q}|\nabla\eta|^2 + \int Q(\Delta\eta)^2 + \int |\nabla(\Delta\eta)|^2 \\
& \lesssim \|\varepsilon\|_{H_Q^2}^2 + \int |\nabla(\Delta\eta)|^2 + \int \frac{|\Delta\eta|^2}{1+r^4} + \int \frac{|\nabla\eta|^2}{1+r^6} \lesssim \|(\varepsilon, \eta)\|_{\mathcal{E}}^2,
\end{aligned}$$

and the continuity is proved.

Step 2 : Adjunction :

To prove rigorously the formal adjoint (2.22), and thus (2.24), we must justify now the integration by parts. For this purpose, let's begin to prove that both integrals are absolutely convergent. First :

$$\begin{aligned}
|\langle \mathcal{L}\varepsilon, \tilde{\varepsilon} \rangle| & \lesssim \int |\mathcal{L}^{(1)}(\varepsilon, \eta)| |\tilde{\varepsilon}| + \int |\partial_r(\Delta\eta - \varepsilon)| |\partial_r\tilde{\eta}| \\
& \lesssim \|\mathcal{L}^{(1)}(\varepsilon, \eta)\|_{L_Q^2} \|\tilde{\varepsilon}\|_{L^2} + \|\mathcal{L}^{(2)}(\varepsilon, \eta)\|_{\dot{H}^1} \|\tilde{\eta}\|_{\dot{H}^1} \\
& \lesssim \|(\varepsilon, \eta)\|_{\mathcal{E}} \|(\tilde{\varepsilon}, \tilde{\eta})\|_{L^2 \times \dot{H}^1}
\end{aligned}$$

Now, using that $\nabla\phi_{\nabla.u} = u$, we obtain using Cauchy-Schwarz

$$\begin{aligned}
|\langle \varepsilon, \mathcal{L}^*\tilde{\varepsilon} \rangle| & = \left| \int \varepsilon \left(\frac{\nabla \cdot \{Q\nabla\tilde{\varepsilon}\}}{Q} + \Delta\tilde{\eta} \right) + \int \nabla\eta \nabla (\Delta\tilde{\eta} - \phi_{\nabla \cdot \{Q\nabla\tilde{\varepsilon}\}}) \right| \\
& \lesssim \|\varepsilon\|_{L^2} \|\Delta\tilde{\eta}\|_{L^2} + \int |\varepsilon| \left[|\Delta\tilde{\varepsilon}| + \frac{|\nabla Q|}{Q} |\nabla\tilde{\varepsilon}| \right] + \int \nabla\eta (\nabla\Delta\tilde{\eta} - \tilde{\varepsilon}) \\
& \lesssim \|(\varepsilon, \eta)\|_{\mathcal{E}} \|(\tilde{\varepsilon}, \tilde{\eta})\|_{\mathcal{E}}
\end{aligned}$$

The integrals being absolutely convergent, we have thus :

$$\langle \mathcal{L}\varepsilon, \tilde{\varepsilon} \rangle = \lim_{R \rightarrow +\infty} \int_0^R \left\{ \mathcal{L}^{(1)}(\varepsilon, \eta)\tilde{\varepsilon} + \nabla\mathcal{L}^{(2)}(\varepsilon, \eta)\nabla\tilde{\eta} \right\} \quad (2.28)$$

Using the radial coordinates, we can rewrite the last integral by:

$$\begin{aligned}
& \int_0^R \left\{ \mathcal{L}^{(1)}(\varepsilon, \eta)\tilde{\varepsilon} + \nabla\mathcal{L}^{(2)}(\varepsilon, \eta)\nabla\tilde{\eta} \right\} \\
& = \int_0^R \left\{ \partial_r \left(rQ\partial_r\mathcal{M}^{(1)}(\varepsilon, \eta) \right) \tilde{\varepsilon} + r\partial_r \left[\frac{1}{r}\partial_r \left(r\partial_r\mathcal{M}^{(2)}(\varepsilon, \eta) \right) \right] \partial_r\tilde{\eta} \right\} dr \\
& = - \int_0^R \left\{ rQ\partial_r\mathcal{M}^{(1)}(\varepsilon, \eta)\partial_r\tilde{\varepsilon} + \partial_r \left(r\partial_r\mathcal{M}^{(2)}(\varepsilon, \eta) \right) \frac{1}{r}\partial_r \left(r\partial_r\tilde{\eta} \right) \right\} dr \\
& + \left[rQ\tilde{\varepsilon}\partial_r\mathcal{M}^{(1)}(\varepsilon, \eta) + \partial_r \left(r\partial_r\mathcal{M}^{(2)}(\varepsilon, \eta) \right) \partial_r\tilde{\eta} \right]_0^R \\
& = \int_0^R \left\{ \mathcal{M}^{(1)}(\varepsilon, \eta) \frac{1}{r}\partial_r \left(rQ\partial_r\tilde{\varepsilon} \right) + \partial_r\mathcal{M}^{(2)}(\varepsilon, \eta)\partial_r \left[\frac{1}{r}\partial_r \left(r\partial_r\tilde{\eta} \right) \right] \right\} r dr \quad (2.29)
\end{aligned}$$

$$+ \left[rQ\tilde{\varepsilon}\partial_r\mathcal{M}^{(1)}(\varepsilon, \eta) + \partial_r \left(r\partial_r\mathcal{M}^{(2)}(\varepsilon, \eta) \right) \partial_r\tilde{\eta} \right]_0^R \quad (2.30)$$

$$- \left[rQ\mathcal{M}^{(1)}(\varepsilon, \eta)\partial_r\tilde{\varepsilon} + \partial_r\mathcal{M}^{(2)}(\varepsilon, \eta)\partial_r \left(r\partial_r\tilde{\eta} \right) \right]_0^R \quad (2.31)$$

Using the smoothness of (ε, η) and of $(\tilde{\varepsilon}, \tilde{\eta})$, the terms (2.30) and (2.31) cancel at the origin. Now, there exists a sequence $R_n \rightarrow +\infty$ such that:

$$\begin{aligned} & \left[rQ\tilde{\varepsilon}\partial_r\mathcal{M}^{(1)}(\varepsilon, \eta) + \partial_r \left(r\partial_r\mathcal{M}^{(2)}(\varepsilon, \eta) \right) \partial_r\tilde{\eta} \right]_0^{R_n} \\ & - \left[rQ\mathcal{M}^{(1)}(\varepsilon, \eta)\partial_r\tilde{\varepsilon} + \partial_r\mathcal{M}^{(2)}(\varepsilon, \eta)\partial_r \left(r\partial_r\tilde{\eta} \right) \right]_0^{R_n} \xrightarrow{n \rightarrow +\infty} 0. \end{aligned} \quad (2.32)$$

Indeed, we estimate from Cauchy-Schwarz and the Hardy bound (A.2) :

$$\begin{aligned} \left| \int Q\partial_r\tilde{\varepsilon} \left(\frac{\varepsilon}{Q} + \eta \right) \right| & \lesssim \left| \int \varepsilon\partial_r\tilde{\varepsilon} \right| + \left| \int Q\eta\partial_r\tilde{\varepsilon} \right| \\ & \lesssim \|\varepsilon\|_{L^2} \|\partial_r\tilde{\varepsilon}\|_{L^2} + \left| \int Q\eta^2 \right|^{\frac{1}{2}} \left| \int Q|\partial_r\tilde{\varepsilon}|^2 \right|^{\frac{1}{2}} \\ & \lesssim \|\varepsilon\|_{L^2} \|\partial_r\tilde{\varepsilon}\|_{L^2} + \|\partial_r\eta\|_{L^2} \|\partial_r\tilde{\varepsilon}\|_{L^2} < +\infty \end{aligned}$$

Moreover :

$$\begin{aligned} \left| \int Q\tilde{\varepsilon}\partial_r \left(\frac{\varepsilon}{Q} + \eta \right) \right| & \lesssim \left| \int \tilde{\varepsilon} (\partial_r\varepsilon + \varepsilon\phi_Q + Q\partial_r\eta) \right| \\ & \lesssim \|\tilde{\varepsilon}\|_{L^2} \left(\left\| \frac{\partial_r\varepsilon}{1+r} \right\|_{L^2_Q} + \|\varepsilon\|_{L^2} + \left\| \frac{\partial_r\eta}{1+y^4} \right\|_{L^2} \right) \end{aligned}$$

Now, with Cauchy-Schwarz

$$\left| \int \partial_r\tilde{\eta} (\Delta\eta - \varepsilon) \right| \lesssim \|\partial_r\tilde{\eta}\|_{L^2} (\|\varepsilon\|_{L^2} + \|\Delta\eta\|_{L^2})$$

Finally,

$$\left| \int \Delta\tilde{\eta} (\partial_r\eta - \partial_r\phi_\varepsilon) \right| \lesssim (\|\partial_r\eta\|_{L^\infty} + \|\partial_r\phi_\varepsilon\|_{L^\infty}) \|\Delta\tilde{\eta}\|_{L^1} \lesssim (\|\Delta\eta\|_{L^2} + \|\varepsilon\|_{L^2}) \|(\tilde{\varepsilon}, \tilde{\eta})\|_\varepsilon$$

This concludes the proof of (2.2). We have proved for the moment that:

$$\left\langle \mathcal{L} \begin{vmatrix} \varepsilon \\ \eta \end{vmatrix}, \begin{vmatrix} \tilde{\varepsilon} \\ \tilde{\eta} \end{vmatrix} \right\rangle = \left\langle \mathcal{M} \begin{vmatrix} \varepsilon \\ \eta \end{vmatrix}, \begin{vmatrix} \nabla \cdot \{Q\nabla\tilde{\varepsilon}\} \\ \Delta\tilde{\eta} \end{vmatrix} \right\rangle$$

With the same method, the proof of the self-adjointness of \mathcal{M} here takes any difficulty, and is left to the reader. This yields (2.24).

Step 3 : Algebraic identities.

The identity (2.6) yields directly (2.25). Now, $\forall c \in \mathbb{R}$

$$\mathcal{L}^*(1, c) = \mathcal{M} \begin{vmatrix} \nabla \cdot \{Q\nabla 1\} \\ \Delta c \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}. \quad (2.33)$$

For the last algebraic identity, we use:

$$\frac{1}{r}\partial_r \left(r\partial_r \left[-2 \int_0^r \frac{\log(1+\tau^2)d\tau}{\tau} \right] \right) = \phi_{\Lambda Q}.$$

Thus,

$$\mathcal{L}^* \left(r^2, -4 \int_0^r \frac{\log(1+\tau^2)d\tau}{\tau} \right) = \mathcal{M} \begin{vmatrix} \nabla \cdot \{Q\nabla r^2\} \\ \Delta \left(-2 \int_0^r \frac{\log(1+\tau^2)d\tau}{\tau} \right) \end{vmatrix} = \mathcal{M} \begin{vmatrix} 2\Lambda Q \\ 2\phi_{\Lambda Q} \end{vmatrix} = \begin{vmatrix} -4 \\ 0 \end{vmatrix}.$$

Step 4 : Vanishing average

Remark that the integral (2.27) is absolutely convergent. Indeed, let $c \in \mathbb{R}$.

$$\left| \left\langle \mathcal{L}\varepsilon, \begin{vmatrix} 1 \\ c \end{vmatrix} \right\rangle \right| \lesssim \left| \int \mathcal{L}^{(1)}(\varepsilon, \eta) \right| \lesssim \int \frac{\mathcal{L}^{(1)}(\varepsilon, \eta)^2}{Q} \lesssim \|(\varepsilon, \eta)\|_{\mathcal{E}}^2$$

Hence,

$$\int \mathcal{L}^{(1)}(\varepsilon, \eta) = \lim_{R \rightarrow +\infty} \int_0^R \mathcal{L}^{(1)}(\varepsilon, \eta)$$

But:

$$\int_0^R \mathcal{L}^{(1)}(\varepsilon, \eta) = \left[rQ\partial_r \left(\frac{\varepsilon}{Q} + \eta \right) \right]_0^R$$

The last term cancels at the origin. Now, using that:

$$\int \left| Q\partial_r \left(\frac{\varepsilon}{Q} + \eta \right) \right|^2 \lesssim \int [|\partial_r \varepsilon| + |\varepsilon|^2 \phi_Q^2 + Q|\partial_r \eta|^2] < +\infty,$$

there exists a sequence $R_n \rightarrow +\infty$ such that:

$$\left\{ Q\partial_r \left(\frac{\varepsilon}{Q} + \eta \right) \right\} (R_n) = o\left(\frac{1}{R_n^2} \right)$$

Thus,

$$\int \mathcal{L}^{(1)}(\varepsilon, \eta) = \lim_{R_n \rightarrow +\infty} \int_0^{R_n} \mathcal{L}^{(1)}(\varepsilon, \eta) = R_n \left\{ Q\partial_r \left(\frac{\varepsilon}{Q} + \eta \right) \right\} (R_n) \rightarrow 0.$$

This concludes the proof of the Lemma 2.3. \square

In the last Lemma, we have exhibited the kernel of the operator \mathcal{L}^* , and we have seen that the elements of this kernel have irrelevant growth. Therefore, we shall introduce in the following Lemma an enough good approximation of this kernel, defining both directions Φ_M and $\mathcal{L}^*\Phi_M$. It is an enough good approximation in the sense that we prove in the Proposition 2.5 that if ε is orthogonal to this directions, then the operator \mathcal{L} is coercive.

Moreover, we must anticipate the construction of the approximate profile, and we define $\mathbf{T}_1 = \begin{vmatrix} T_1 \\ S_1 \end{vmatrix}$ such that $\mathcal{L}\mathbf{T}_1 = \Lambda\mathbf{Q}$. Thus, we have the following bounds:

$$T_1(r) \lesssim \frac{1}{1+r^2}, \quad \partial_r S_1(r) \lesssim \frac{r}{1+r^2}.$$

Lemma 2.4 (On the direction Φ_M). *Given $M \geq M_0 > 1$ large enough, we define the directions: Π*

$$\Phi_{0,M} = \begin{vmatrix} \Phi_{0,M} \\ \Pi_{0,M} \end{vmatrix} = \begin{vmatrix} \chi_M r^2 \\ -4 \int_0^r \frac{\log(1+\tau^2)}{\tau} \chi_M d\tau \end{vmatrix}, \quad (2.34)$$

$$\Phi_M(y) = \Phi_{0,M} + c_M \mathcal{L}^*(\Phi_{0,M}), \quad c_M = -\frac{\langle \Phi_{0,M}, \mathbf{T}_1 \rangle}{\langle \Phi_{0,M}, \Lambda\mathbf{Q} \rangle}, \quad (2.35)$$

Then:

(i) Estimate on Φ_M :

$$\Phi_M(\mathbf{r}) = \Phi_{0,M} - \chi_M \begin{vmatrix} 4c_M \\ 0 \end{vmatrix} + \frac{M^2}{\log M} O \left(\begin{vmatrix} \mathbf{1}_{M \leq r \leq 2M} \\ \frac{1}{1+r^2} \mathbf{1}_{r \leq 2M} + \frac{\log M}{M^2} \mathbf{1}_{M \leq r \leq 2M} \end{vmatrix} \right), \quad (2.36)$$

$$\langle \Phi_M, \mathbf{T}_1 \rangle = 0, \quad \langle \Phi_M, \Lambda\mathbf{Q} \rangle = -(32\pi)\log M + O_{M \rightarrow +\infty}(1), \quad (2.37)$$

(ii) Estimate on scalar products: $\forall(\varepsilon, \eta)$ such that $(\varepsilon, \nabla\eta) \in L^1 \times L^1 \left(\frac{dr}{1+r^3} \right)$,

$$|\langle \varepsilon, \Phi_M \rangle| \lesssim \int_{r \leq 2M} \left\{ (1+r^2)|\varepsilon| + \frac{|\partial_r \eta| \log(1+r^2)}{r} \right\} + \frac{M^2}{\log M} \left[|(\varepsilon, 1)| + \int_{r \geq M} |\varepsilon| \right], \quad (2.38)$$

$$|\langle \varepsilon, \mathcal{L}^* \Phi_{0,M} \rangle| \lesssim \int_{r \leq 2M} |\varepsilon| + \log M \int_{M \leq r \leq 2M} \frac{|\partial_r \eta|}{1+r^3}, \quad (2.39)$$

$$|(\varepsilon, \mathcal{L}^* \Phi_M)| \lesssim \int_{r \leq 2M} |\varepsilon| + \log M \int_{M \leq r \leq 2M} \frac{|\partial_r \eta|}{1+r^3} + \frac{M^2}{\log M} \int_{r \geq M} \frac{|\varepsilon|}{1+r^2}, \quad (2.40)$$

$$|(\varepsilon, \mathcal{L}^* \Phi_M) + 4(\varepsilon, 1)| + |(\varepsilon, \mathcal{L}^* \Phi_{0,M}) + 4(\varepsilon, 1)| \lesssim \int_{r \geq M} |\varepsilon| + \log M \int_{M \leq r \leq 2M} \frac{|\partial_r \eta|}{1+r^3} \quad (2.41)$$

(ii) Rough bounds: if moreover $\varepsilon \in X_Q$:

$$|\langle \varepsilon, \Phi_M \rangle| + |\langle \varepsilon, \Phi_{0,M} \rangle| \lesssim M \|\varepsilon\|_{X_Q}^2 + \frac{M^2}{\log M} \left| \int \varepsilon \right| \quad (2.42)$$

$$|(\varepsilon, \mathcal{L}^* \Phi_M)| \lesssim M \left(\|\varepsilon\|_{L^2} + \left\| \frac{\partial_r \eta}{1+r^3} \right\|_{L^2} \right). \quad (2.43)$$

Proof. Step 1 : Proof of estimate on Φ_M .

We begin to compute $\mathcal{L}^*(\Phi_{0,M})$. We let:

$$\mathcal{L}^* \begin{vmatrix} \Phi_{0,M} \\ \Pi_{0,M} \end{vmatrix} = \mathcal{M} \begin{vmatrix} r_{1,M} \\ r_{2,M} \end{vmatrix}, \text{ with } \begin{vmatrix} r_{1,M} \\ r_{2,M} \end{vmatrix} = \begin{vmatrix} \nabla \cdot (Q \nabla (\Phi_{0,M})) \\ \Delta \Pi_{0,M} \end{vmatrix}. \quad (2.44)$$

Hence,

$$\begin{vmatrix} r_{1,M} \\ r_{2,M} \end{vmatrix} = 2\Lambda Q \chi_M + \begin{vmatrix} (5rQ + r^2 Q') \chi'_M + r^2 Q \chi''_M \\ -4 \frac{\log(1+r^2)}{r} \chi'_M \end{vmatrix} \quad (2.45)$$

Moreover, as $\int r_{1,M} = \int \nabla \cdot (Q \nabla (\Phi_{0,M})) = 0$, we have the following expression for the Poisson field of $r_{1,M}$:

$$|\phi_{r_{1,M}}(r)| = \left| \int_r^{+\infty} Q \partial_r (\chi_M r^2) dr \right| \lesssim \frac{1}{1+r^2} \mathbf{1}_{r \leq 2M}. \quad (2.46)$$

By definition

$$\mathcal{L}^* \begin{vmatrix} \Phi_{0,M} \\ \Pi_{0,M} \end{vmatrix} = \begin{vmatrix} \frac{r_{1,M}}{Q} + r_{2,M} \\ r_{2,M} - \phi_{r_{1,M}} \end{vmatrix},$$

and thus, the decay of $Q(r) \lesssim \frac{1}{1+r^4}$ together (2.45) and (2.46) yield:

$$\mathcal{L}^* \begin{vmatrix} \Phi_{0,M} \\ \Pi_{0,M} \end{vmatrix} = \begin{vmatrix} -4\chi_M + O(\mathbf{1}_{M \leq r \leq 2M}) \\ \frac{1}{1+r^2} O(\mathbf{1}_{r \leq 2M}) + \frac{\log M}{M^2} O(\mathbf{1}_{M \leq r \leq 2M}) \end{vmatrix}. \quad (2.47)$$

Moreover we have the following cancellation:

$$\left| \partial_r \left((\mathcal{L}^*)^{(2)} (\Phi_{0,M}, \Pi_{0,M}) \right) \right| = |\partial_r(r_{2,M}) - \partial_r(\phi_{r_{1,M}})| \lesssim \frac{\log M}{M^3} \mathbf{1}_{M \leq y \leq 2M} \quad (2.48)$$

Indeed:

$$\partial_r(r_{2,M}) = 2\partial_r(\phi_{\Lambda Q})\chi_M + O\left(\frac{\log M}{M^3} \mathbf{1}_{M \leq y \leq 2M}\right)$$

Moreover

$$\partial_r(\phi_{r_{1,M}}) = Q \partial_r(r^2 \chi_M) = 2rQ \chi_M + O\left(\frac{1}{M^3} \mathbf{1}_{M \leq y \leq 2M}\right)$$

The identity $\partial_r(\phi_{\Lambda Q}) = rQ$ concludes the proof of (2.48). Now, remark that $\Lambda \mathbf{Q} \in \mathcal{E}$, and using the Lemma (2.3), we obtain

$$\langle \mathcal{L}^* \Phi_{0,M}, \Lambda \mathbf{Q} \rangle = \langle \Phi_{0,M}, \mathcal{L} \Lambda \mathbf{Q} \rangle = 0$$

and so

$$\langle \Phi_{0,M}, \Lambda \mathbf{Q} \rangle = \langle \Phi_M, \Lambda \mathbf{Q} \rangle.$$

Now:

$$\begin{aligned} \int \chi_M r^2 \Lambda Q(r) &= 2\pi \int \chi_M r^2 \partial_r (r^2 Q) dr = -2\pi \int r^2 Q \partial_r (\chi_M r^2) dr \\ &= -2\pi \int Q r^3 \chi_M dr + O\left(\frac{2\pi}{M} \int_M^{2M} Q r^4 dr\right) = -32\pi \log M + O(1). \end{aligned}$$

Moreover

$$\int \partial_r \Pi_{0,M} \partial_r \phi_{\Lambda Q} = \int \frac{16r \log(1+r^2)}{(1+r^2)^2} dr < +\infty.$$

Finally, we obtain:

$$\langle \Phi_{0,M}, \Lambda \mathbf{Q} \rangle = \langle \Phi_M, \Lambda \mathbf{Q} \rangle = -32\pi \log M + O(1). \quad (2.49)$$

The compact support of $\Phi_{0,M}$ and $\partial_r \Pi_{0,M}$ together the decay of T_1 and $\partial_r S_1$ easily justify that:

$$\langle \mathcal{L}^* \Phi_{0,M}, \mathbf{T}_1 \rangle = \langle \Phi_{0,M}, \mathcal{L} \mathbf{T}_1 \rangle = \langle \Phi_{0,M}, \Lambda \mathbf{Q} \rangle$$

This yields (2.37). Using $|T_1(r)| \lesssim \frac{1}{1+r^2}$ and $|\partial_r S_1(r)| \lesssim \frac{r}{1+r^2}$, we can compute

$$|\langle \Phi_{0,M}, \mathbf{T}_1 \rangle| \lesssim \int \frac{r^2}{1+r^2} \chi_M + \int \frac{\log(1+r^2)}{1+r^2} \chi_M \lesssim M^2.$$

With (2.49), we have the upper bound:

$$|c_M| \lesssim \frac{M^2}{\log M} \quad (2.50)$$

which concludes the proof of (i).

Step 2 : Proof of the estimate on the scalar product.

First, using the definition (2.34) of the direction $\Phi_{0,M}$:

$$|\langle \varepsilon, \Phi_{0,M} \rangle| \lesssim \int_{y \leq 2M} r^2 |\varepsilon| + \int_{y \leq 2M} \frac{|\partial_r \eta| (\log(1+r^2))}{r}. \quad (2.51)$$

Now, using (2.47) and the cancellation (2.48) :

$$\begin{aligned} \langle \varepsilon, \mathcal{L}^* \Phi_{0,M} \rangle &= \int \varepsilon (-4\chi_M + O(\mathbf{1}_{M \leq r \leq 2M})) + O\left(\int \nabla \eta \frac{\log M}{M^3} \mathbf{1}_{M \leq y \leq 2M}\right) \\ &= -4 \int \varepsilon + O\left(\int_{y \geq M} |\varepsilon| + \log M \int_M^{2M} \frac{|\nabla \eta|}{1+y^3}\right) \end{aligned} \quad (2.52)$$

This result together (2.51), the definition (2.35) of the direction Φ_M , and the bound (2.50) yields (2.38) and (2.39). To finish the proof of the Lemma 2.4, we must estimate $(\mathcal{L}^*)^2(\Phi_M)$. Let

$$(\mathcal{L}^*)^2(\Phi_M) = \mathcal{M} \begin{vmatrix} r_3 \\ r_4 \end{vmatrix} = \mathcal{M} \begin{vmatrix} \nabla \cdot (Q \nabla (\mathcal{M}^{(1)}(r_1, r_2))) \\ \Delta (\mathcal{M}^{(2)}(r_1, r_2)) \end{vmatrix}$$

With (2.47) and (2.48), we obtain the following bound :

$$\begin{aligned} |r_3| &\lesssim \frac{1}{1+y^6} \mathbf{1}_{M \leq y \leq 2M} \\ |r_4| &\lesssim \frac{\log M}{M^4} \mathbf{1}_{M \leq y \leq 2M} \end{aligned}$$

and thus, using that $\partial_r \left(\phi_{\nabla \cdot (Q \nabla (\mathcal{M}^{(1)}(r_1, r_2)))} \right) = Q \partial_r (\mathcal{M}^{(1)}(r_1, r_2))$

$$\begin{aligned} |\mathcal{M}^{(1)}(r_3, r_4)| &\lesssim \frac{1}{1+y^2} \mathbf{1}_{M \leq y \leq 2M} \\ \left| \partial_r \left(\mathcal{M}^{(2)}(r_3, r_4) \right) \right| &\lesssim \frac{\log M}{M^5} \mathbf{1}_{M \leq y \leq 2M} \end{aligned}$$

Hence,

$$\left| \langle \varepsilon, (\mathcal{L}^*)^2(\Phi_M) \rangle \right| \lesssim \int_{M \leq y \leq 2M} \frac{|\varepsilon|}{1+r^2} + \frac{\log M}{M^2} \int_{M \leq y \leq 2M} \frac{|\partial_r \eta|}{1+y^3}$$

With this bound, we obtain any difficulty (2.40) and (2.41). (2.42) and (2.43) respectively come from (2.38) and (2.40) using Cauchy-Schwarz. This concludes the proof of the Lemma 2.4 \square

We are now in position to derive the fundamental coercivity property described in the following Proposition at the heart of our analysis. Moreover, we track the M dependence of constants which is crucial for the derivation of the blow-up speed.

Proposition 2.5 (Coercivity of \mathcal{L}). *There exist universal constants $\delta_0, M_0 > 0$ such that $\forall M \geq M_0$, there exists $\delta(M) > 0$ such that the following holds. Let $\varepsilon \in \mathcal{E}$ with satisfying the following orthogonality conditions:*

$$\langle \varepsilon, \Phi_M \rangle = \langle \varepsilon, \mathcal{L}^* \Phi_M \rangle = 0. \quad (2.53)$$

Then there hold the bounds:

(i) Control of $\mathcal{L}(\varepsilon, \eta)$:

$$\langle \mathcal{M} \mathcal{L}(\varepsilon, \eta), \mathcal{L}(\varepsilon, \eta) \rangle \geq \frac{\delta_0 (\log M)^2}{M^2} \|\mathcal{L}(\varepsilon, \eta)\|_{X_Q}^2, \quad (2.54)$$

(ii) Coercivity of \mathcal{L} :

$$\begin{aligned} \frac{1}{\delta(M)} \|\mathcal{L}(\varepsilon, \eta)\|_{X_Q}^2 &\geq \int (1+r^4) |\Delta \varepsilon|^2 + \int (1+r^2) |\nabla \varepsilon|^2 + \int \varepsilon^2 \\ &+ \int \frac{|\nabla \phi_\varepsilon|^2}{r^2(1+|\log r|)^2} + \int |\nabla(\Delta \eta)|^2 \\ &+ \int \frac{|\Delta \eta|^2}{r^2(1+|\log r|)^2} + \int \frac{|\nabla \eta|^2}{r^2(1+r^2)(1+|\log r|)^2} \end{aligned} \quad (2.55)$$

Proof. Step 1 : Control of $\mathcal{L}\varepsilon$.

Let $\begin{pmatrix} \varepsilon \\ \eta \end{pmatrix} \in \mathcal{E}$. Let

$$\begin{pmatrix} \varepsilon_2 \\ \eta_2 \end{pmatrix} = \mathcal{L} \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix} = \begin{pmatrix} \nabla \cdot (\nabla \varepsilon + \nabla \phi_Q \varepsilon + \nabla \eta) \\ \Delta \eta - \varepsilon \end{pmatrix}.$$

The definition of the energy space \mathcal{E} , and the vanishing average (2.27) of the Lemma 2.3 assure that $\varepsilon_2 \in X_Q$. Moreover, from the choice of orthogonality conditions (2.54), and the adjunction (2.24) :

$$\langle \varepsilon_2, \Phi_M \rangle = \langle \mathcal{L}\varepsilon, \Phi_M \rangle = \langle \varepsilon, \mathcal{L}^* \Phi_M \rangle = 0.$$

In order to know precisely the M-dependence of the constants, we use the following suitable function:

$$\begin{cases} \tilde{\varepsilon}_2 \\ \tilde{\eta}_2 \end{cases} = \begin{cases} \varepsilon_2 \\ \eta_2 \end{cases} - a_1 \Lambda \mathbf{Q} \quad (2.56)$$

with

$$a_1 = \frac{\langle \varepsilon_2, \Lambda \mathbf{Q} \rangle}{\langle \Lambda \mathbf{Q}, \Lambda \mathbf{Q} \rangle} \quad (2.57)$$

which yields :

$$\langle \tilde{\varepsilon}_2, \Lambda \mathbf{Q} \rangle = 0. \quad (2.58)$$

Furthermore, the cancellation

$$\int \Lambda Q = 0$$

together the definition of $\tilde{\varepsilon}_2$ yield that $\tilde{\varepsilon}_2 \in X_Q$. Thus, we can apply the Lemma 2.2, and obtain the bound:

$$\langle \mathcal{M}\tilde{\varepsilon}_2, \tilde{\varepsilon}_2 \rangle \geq \delta_0 \|\tilde{\varepsilon}_2\|_{X_Q}^2 \quad (2.59)$$

Next

$$\langle \mathcal{M}\varepsilon_2, \Lambda \mathbf{Q} \rangle = \langle \varepsilon_2, \mathcal{M}\Lambda \mathbf{Q} \rangle = 0 \quad (2.60)$$

and

$$\langle \mathcal{M}\Lambda \mathbf{Q}, \Lambda \mathbf{Q} \rangle = 0. \quad (2.61)$$

Thus

$$\begin{aligned} \langle \mathcal{M}\tilde{\varepsilon}_2, \tilde{\varepsilon}_2 \rangle &= \langle \mathcal{M}\varepsilon_2 - a_1 \mathcal{M}\Lambda \mathbf{Q}, \varepsilon_2 - a_1 \Lambda \mathbf{Q} \rangle \\ &= \langle \mathcal{M}\varepsilon_2, \varepsilon_2 \rangle - 2a_1 \langle \mathcal{M}\varepsilon_2, \Lambda \mathbf{Q} \rangle + a_1^2 \langle \mathcal{M}\Lambda \mathbf{Q}, \Lambda \mathbf{Q} \rangle \\ &= \langle \mathcal{M}\varepsilon_2, \varepsilon_2 \rangle \end{aligned} \quad (2.62)$$

Finally, we use the orthogonality condition on ε_2 , and the bound (2.42) to estimate:

$$\begin{aligned} |a_1| &= \left| \frac{\langle \tilde{\varepsilon}_2, \Phi_M \rangle}{\langle \Lambda \mathbf{Q}, \Phi_M \rangle} \right| \lesssim \frac{1}{|\log M|} \left[M \|\tilde{\varepsilon}_2\|_{X_Q} + \frac{M^2}{\log M} \left| \int \tilde{\varepsilon}_2 \right| \right] \\ &\lesssim \frac{M}{\log M} \|\tilde{\varepsilon}_2\|_{X_Q}, \end{aligned}$$

and thus

$$\|\varepsilon_2\|_{X_Q} \lesssim \frac{M}{\log M} \|\tilde{\varepsilon}_2\|_{X_Q} \quad (2.63)$$

which together (2.59) and (2.62) concludes the proof of (2.54).

Step 2 : Subcoercivity of \mathcal{L}

In order to prove (2.55), we begin to prove the subcoercivity estimate, which is describe in the following inequality, based on two dimensional Hardy inequalities and a very good knowledge of the structure of the operator. We will be then in position to prove the coercivity of \mathcal{L} under orthogonality conditions (2.53) from a

compactness argument. Let $\varepsilon \in X_Q$. Then:

$$\begin{aligned}
& \int \frac{(\mathcal{L}^{(1)}(\varepsilon, \eta))^2}{Q} + \int |\nabla \mathcal{L}^{(2)}(\varepsilon, \eta)|^2 \\
& \gtrsim \int (1+r^4)(\Delta\varepsilon)^2 + \int \left[\frac{1}{r^2(1+|\log r|)^2} + r^2 \right] |\nabla\varepsilon|^2 + \int \varepsilon^2 + \int |\nabla(\Delta\eta)|^2 \\
& + \int \frac{|\Delta\eta|^2}{r^2(1+|\log r|)^2} + \int \frac{|\nabla\eta|^2}{r^2(1+r^2)(1+|\log r|)^2} + \int \frac{\eta^2}{(1+r^6)(1+|\log r|)^2} \\
& - \int \frac{\varepsilon^2}{1+r^2} - \int |\nabla\varepsilon|^2 - \int \frac{|\Delta\eta|^2}{1+r^4} - \int \frac{|\nabla\eta|^2}{1+r^6}. \tag{2.64}
\end{aligned}$$

Proof of (2.64) :

Using the explicit expression of $\mathcal{L}^{(1)}$:

$$\begin{aligned}
\int \frac{|\mathcal{L}^{(1)}(\varepsilon, \eta)|^2}{Q} & \gtrsim \int \frac{(\Delta\varepsilon + \nabla\phi_Q \cdot \nabla\varepsilon)^2}{Q} - \int \frac{|\nabla Q|^2}{Q} |\nabla\eta|^2 - \int Q(\varepsilon^2 + |\Delta\eta|^2) \\
& \gtrsim \int \frac{(\Delta\varepsilon + \nabla\phi_Q \cdot \nabla\varepsilon)^2}{Q} - \int \frac{\varepsilon^2 + |\Delta\eta|^2}{1+r^4} - \int \frac{|\nabla\eta|^2}{1+r^6}. \tag{2.65}
\end{aligned}$$

The first term of RHS requires a carefully compute. Let's develop it:

$$\int \frac{(\Delta\varepsilon + \nabla\phi_Q \cdot \nabla\varepsilon)^2}{Q} = \int \frac{(\Delta\varepsilon)^2}{Q} + \frac{(\partial_r\phi_Q \partial_r\varepsilon)^2}{Q} + \int \frac{2}{Q} \Delta\varepsilon \nabla\phi_Q \cdot \nabla\varepsilon.$$

Now we observe that :

$$\frac{2\nabla\phi_Q}{Q} = -2\frac{\nabla Q}{Q^2} = 2\nabla\psi, \quad \psi = \frac{1}{Q}.$$

With the classical Pohozaev integration by parts formula:

$$\begin{aligned}
2 \int \Delta\varepsilon \partial_r\psi \partial_r\varepsilon & = 2 \int \partial_r(r\partial_r\varepsilon) \partial_r\psi \partial_r\varepsilon dr \\
& = - \int (r\partial_r\varepsilon)^2 \partial_r \left(\frac{\partial_r\psi}{r} \right) dr = - \int (\partial_r\varepsilon)^2 \left[\partial_r^2\psi - \frac{\partial_r\psi}{r} \right].
\end{aligned}$$

Moreover, we have the following Taylor series for $r \gg 1$:

$$\begin{aligned}
\psi(r) & = \frac{1}{Q} = \frac{r^4}{8} + O(r^2), \quad \partial_r^2\psi - \frac{\partial_r\psi}{r} = r^2 + O(1), \\
\phi'_Q(r) & = \frac{1}{r} \int_0^r Q(\tau)\tau d\tau = \frac{4}{r} + O\left(\frac{1}{r^3}\right), \quad \frac{(\partial_r\phi_Q)^2}{Q} = 2r^2 + O(1).
\end{aligned}$$

Thus:

$$\int \frac{(\Delta\varepsilon + \nabla\phi_Q \cdot \nabla\varepsilon)^2}{Q} \gtrsim \int (1+r^4)(\Delta\varepsilon)^2 + \int (2r^2 - r^2)(\partial_r\varepsilon)^2 - \int |\nabla\varepsilon|^2.$$

The above inequality together (2.65) yields the lower bound:

$$\int \frac{|\mathcal{L}^{(1)}(\varepsilon, \eta)|^2}{Q} \gtrsim \int (1+r^4)(\Delta\varepsilon)^2 + \int r^2 |\nabla\varepsilon|^2 - \int \frac{\varepsilon^2 + |\Delta\eta|^2}{1+r^4} - \int |\nabla\varepsilon|^2 - \int \frac{|\nabla\eta|^2}{1+r^6} \tag{2.66}$$

Now,

$$\int |\nabla \mathcal{L}^{(2)}(\varepsilon, \eta)|^2 \gtrsim \int |\nabla(\Delta\eta)|^2 - \int |\nabla\varepsilon|^2. \tag{2.67}$$

Injecting the Hardy bounds of the Lemma (A.1) in (2.66) and (2.67) conclude the proof of (2.64).

Step 3 : Coercivity of \mathcal{L} :

Let's prove now (2.55). In fact, we prove it for $(\varepsilon, \eta) \in \mathcal{C}_c^\infty(\mathbb{R})^2$, such that $\varepsilon(0) = \Delta\eta(0)$. Using the continuity of \mathcal{L} and a standard argument of density, it is enough to obtain (2.55).

In the same way as the proof of the coercivity of \mathcal{M} , we argue by contradiction. Let a sequence $\varepsilon_p = (\varepsilon_p, \eta_p) \in \mathcal{C}_c^\infty(\mathbb{R})^2$ such that :

- Condition at the origin : $\varepsilon_p(0) = \Delta\eta_p(0)$.
- Orthogonality condition : $\langle \varepsilon_p, \Phi_M \rangle = \langle \varepsilon_p, L^* \Phi_M \rangle = 0$.
- Normalization :

$$\begin{aligned} & \int (1+r^4)(\Delta\varepsilon_p)^2 + \int \left[\frac{1}{r^2(1+|\log r|)^2} + r^2 \right] |\nabla\varepsilon_p|^2 + \int \varepsilon_p^2 + \int |\nabla(\Delta\eta_p)|^2 \\ + & \int \frac{|\Delta\eta_p|^2}{r^2(1+|\log r|)^2} + \int \frac{|\nabla\eta_p|^2}{r^2(1+r^2)(1+|\log r|)^2} + \int \frac{\eta_p^2}{(1+r^6)(1+|\log r|)^2} = 1 \end{aligned}$$

- Control of the X_Q -norm : $\|\varepsilon_p\|^2 \leq \frac{1}{p}$.

The normalization condition implies that the sequence ε_p is uniformly bound in $H_{loc}^2 \times H_{loc}^3$. Hence, we may extract up to a subsequence such that it weakly converges in $H_{loc}^2 \times H_{loc}^3$ to $(\varepsilon_\infty, \eta_\infty)$ with the orthogonality conditions :

$$\langle \varepsilon_\infty, \Phi_M \rangle = \langle \varepsilon_\infty, L^* \Phi_M \rangle = 0. \quad (2.68)$$

Moreover, the control of the X_Q -norm implies that:

$$\mathcal{L}(\varepsilon_\infty, \eta_\infty) = \begin{vmatrix} \frac{1}{r} \partial_r \left(rQ \partial_r \left(\frac{\varepsilon_\infty}{Q} + \eta_\infty \right) \right) \\ \Delta\eta_\infty - \varepsilon_\infty \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}. \quad (2.69)$$

As we have for all p , $\phi_{\Delta\eta_p} = \eta_p$, it's clear that $\phi_{\Delta\eta_\infty} = \eta_\infty$. Now, (2.69) implies that

$$rQ \partial_r \left(\frac{\varepsilon_\infty}{Q} + \eta_\infty \right) = c, \quad c \in \mathbb{R}. \quad (2.70)$$

But

$$\left| rQ \partial_r \left(\frac{\varepsilon_\infty}{Q} + \eta_\infty \right) \right| \lesssim r |\partial_r \varepsilon| + |\varepsilon| + \frac{|\partial_r \eta|}{1+r^3}.$$

With the normalization condition, and the weakly convergence, we obtain the upper bound :

$$\begin{aligned} & \int (1+r^4)(\Delta\varepsilon_\infty)^2 + \int \left[\frac{1}{r^2(1+|\log r|)^2} + r^2 \right] |\nabla\varepsilon_\infty|^2 + \int \varepsilon_\infty^2 + \int |\nabla(\Delta\eta_\infty)|^2 \\ + & \int \frac{|\Delta\eta_\infty|^2}{r^2(1+|\log r|)^2} + \int \frac{|\nabla\eta_\infty|^2}{r^2(1+r^2)(1+|\log r|)^2} + \int \frac{\eta_\infty^2}{(1+r^6)(1+|\log r|)^2} \leq 1 \end{aligned}$$

Hence, the constant c of (2.70) equals 0. The generalized kernel (2.7) of the operator \mathcal{M} of the Lemma 2.1 ensures that $\varepsilon_\infty \in \text{Span}(\mathbf{A}\mathbf{Q})$, and the orthogonality conditions (2.68) implies that $\varepsilon_\infty = 0$.

Now, the control of the X_Q -norm together the subcoercivity lower bound (2.64) yield the non degeneracy :

$$\int \frac{\varepsilon_\infty^2}{1+r^2} + \int |\nabla\varepsilon_\infty|^2 + \int \frac{|\Delta\eta_\infty|^2}{1+r^4} + \int \frac{|\nabla\eta_\infty|^2}{1+r^6} > 0.$$

which contradicts $\varepsilon_\infty = 0$. This concludes the proof of the Proposition 2.5. \square

3. Construction of approximate solution

The purpose of this section is to obtain an approximate blow-up solution of (1.2). More precisely, the constructed solution will contain the main qualitative informations on the dynamics of the singularity formation. We will measure the made error by suitable quantities which will allow us to prove thereafter the smallness of the gap between this approximate solution and the exact solution, in the sense that this gap doesn't perturb the blow-up dynamic found in this section.

3.1. On the rescaled variables. First, we can remark that if (u, v) is solution of (1.2), then $(u, v+c)$ with $c \in \mathbb{R}$ too. So, before introducing the rescaled variables, the second equation of the system (1.2) are being gone through the operator gradient. Hence, (1.2) becomes:

$$(PKS') \begin{cases} \partial_t u = \nabla \cdot (\nabla u + u \nabla v), \\ \partial_t \nabla v = \nabla \Delta v - \nabla u, \\ u|_{t=0} = u_0 > 0, \\ \nabla v|_{t=0} = \nabla v_0 \end{cases} \quad (t, x) \in (\mathbb{R} \times \mathbb{R}^2) \quad (3.1)$$

Moreover, we have the following scaling invariance: if $(u(t, r), \nabla v(t, r))$ solves (3.1), then so does $(u_\lambda, \nabla v_\lambda) = (\frac{1}{\lambda^2} u(\frac{t}{\lambda^2}, \frac{r}{\lambda}), \frac{1}{\lambda} \nabla v(\frac{t}{\lambda^2}, \frac{r}{\lambda}))$ for $\lambda > 0$.

Now, let $\lambda(t)$ a regular non negative function. Let the rescaled variables:

$$s = \int_0^t \frac{d\tau}{\lambda^2(\tau)}, \quad y = \frac{r}{\lambda(t)} \quad (3.2)$$

As we look for a slower blow up than the self similar regime, ie $\lambda(t) < \sqrt{T-t}$, noticing T the blow-up time. We can remark that $s(t)$ is a bijection between $[0, T[$ and \mathbb{R}^+ . With this new variables (3.1) becomes

$$\partial_s \begin{vmatrix} u \\ \nabla v \end{vmatrix} - \frac{\lambda_s}{\lambda} \Lambda \begin{vmatrix} u \\ \nabla v \end{vmatrix} = \begin{vmatrix} \nabla \cdot (\nabla u + u \nabla v) \\ \nabla (\Delta v - u) \end{vmatrix} \quad (3.3)$$

where we recall that Λ is the scaling operator define by:

$$\Lambda \begin{vmatrix} f \\ \nabla g \end{vmatrix} = \begin{vmatrix} \Lambda^{(1)} f \\ \Lambda^{(2)} \nabla g \end{vmatrix} = \begin{vmatrix} 2f + y \cdot \nabla f \\ \nabla g + y \cdot \nabla^2 g \end{vmatrix} = \begin{vmatrix} \nabla \cdot (yf) \\ y \cdot \Delta g \end{vmatrix} \quad (3.4)$$

3.2. Introduction of the main tools. First, we let

$$b(s) = -\frac{\lambda_s}{\lambda}. \quad (3.5)$$

In the blow-up regime described in the Theorem 1.1, we can see that b is a very small non negative function. In fact, in the next sections, we relax this constraint to obtain two independent parameters of modulation b and λ , which we allow us to fix two orthogonality conditions for the gap between the constructed approximate solution and the exact solution. We saw in the last section the importance of this to have coercivity properties for the operators \mathcal{L} and \mathcal{M} .

Now, we look for an approximate solution \mathbf{Q}_b of (3.1) close to \mathbf{Q} in the form :

$$\mathbf{Q}_b = \begin{vmatrix} Q_b(r) \\ P_b(r) \end{vmatrix} = \mathbf{Q} + b \begin{vmatrix} T_1(r) \\ S_1(r) \end{vmatrix} + b^2 \begin{vmatrix} T_2(b, r) \\ S_2(b, r) \end{vmatrix} = \mathbf{Q} + \mathbf{T}_b = \mathbf{Q} + \begin{vmatrix} \alpha_b(r) \\ \gamma_b(r) \end{vmatrix} \quad (3.6)$$

where \mathbf{T}_1 and \mathbf{T}_2 are profiles independent on rescaled time s , which we will determine. The error Ψ_b associated to \mathbf{Q}_b is defined according to the formula :

$$\begin{vmatrix} \Psi_b^{(1)} \\ \nabla \Psi_b^{(2)} \end{vmatrix} = \begin{vmatrix} \nabla \cdot (\nabla Q_b + Q_b \nabla P_b) \\ \nabla (\Delta P_b - Q_b) \end{vmatrix} - b \Lambda \begin{vmatrix} Q_b(r) \\ \nabla P_b(r) \end{vmatrix} + c_b b^2 \chi_{\frac{B_0}{4}} \begin{vmatrix} T_1 \\ \nabla S_1 \end{vmatrix} \quad (3.7)$$

with c_b given by (3.50). Remark that $\Psi_b^{(2)}$ is defined to within a function of s , which we don't have to determine. The last term of the RHS comes from the radiation which we will use in the construction to improve the size of Ψ_b . This term has to be extracted from the error. Indeed, whereas we will can use the inequality of Cauchy-Schwartz for the terms depending on Ψ_b in the control of the gap between Q_b and the exact solution, we will have to be careful with this last term, and use its particular structure.

Moreover, indicate that the constructed profiles will have pathological growth outside the parabolic zone. Hence, after determining this profiles, we must localize them. We will see in particular that the choice $B_1 = \frac{|\log b|}{\sqrt{b}}$ comes from terms of error due to the localization, which mustn't be bigger than terms of error due to the construction.

Let's introduce the partial mass associated to Q_b

$$m_b(r) = \int_0^r Q_b(\tau) \tau d\tau. \quad (3.8)$$

and the mass partial associated to ΔP_b

$$n_b(r) = \int_0^r \Delta P_b(\tau) \tau d\tau. \quad (3.9)$$

Hence, remarking that

$$\nabla \phi_{Q_b}(r) = \frac{m_b}{r}, \quad Q_b(r) = \frac{m'_b}{r}, \quad \nabla P_b(r) = \frac{n_b}{r}, \quad \text{and} \quad \Delta P_b(r) = \frac{n'_b}{r}, \quad (3.10)$$

we can rewrite (3.25) by

$$\left| \begin{array}{l} \Psi_b^{(1)} \\ \nabla \Psi_b^{(2)} \end{array} \right| = \frac{1}{r} \left| \begin{array}{l} \Phi'_b \\ \Omega_b \end{array} \right| + c_b b^2 \chi_{\frac{B_0}{4}} \left| \begin{array}{l} T_1 \\ \nabla S_1 \end{array} \right| \quad (3.11)$$

where

$$\begin{aligned} \left| \begin{array}{l} \Phi_b \\ \Omega_b \end{array} \right| &= \left| \begin{array}{l} m''_b - \frac{m'_b}{r} + \frac{m'_b n_b}{r^2} - br m'_b \\ (n_b - m_b)'' - \frac{(n_b - m_b)'}{r} - br n'_b \end{array} \right| \\ &= \left| \begin{array}{l} m''_b - \frac{m'_b}{r} + \frac{m'_b m_b}{r} + \frac{m'_b d_b}{r} - br m'_b \\ d''_b - \frac{d'_b}{r} - br n'_b \end{array} \right| \end{aligned} \quad (3.12)$$

with

$$d_b = n_b - m_b. \quad (3.13)$$

We proceed to an expansion on the form :

$$m_b = m_0 + b m_1 + b^2 m_2, \quad n_b = n_0 + b n_1 + b^2 n_2, \quad d_b = d_0 + b d_1 + b^2 d_2 \quad (3.14)$$

with

$$m_0(r) = n_0(r) = \int_0^r Q(y) y dy = \frac{4r^2}{1+r^2}, \quad d_0(r) = 0, \quad (3.15)$$

m_1 and m_2 the partial mass of respectively T_1 and T_2 , n_1 and n_2 the partial mass of respectively ΔS_1 and ΔS_2 . The subject of this section is to determine m_1 , m_2 , n_1 and n_2 , in order to minimize in a suitable sense the size of Ψ_b . So, it will need to find a function u solution of

$$L_0 u = -f, \quad (3.16)$$

where f is a given function, and L_0 the linearized operator close to m_0 of (3.12) given by:

$$L_0 m = -m'' + \left(\frac{1}{r} + \frac{Q'}{Q} \right) m' - Qm.$$

We recall that the basis of solutions to this homogeneous equation is explicit and given by:

$$\psi_0(r) = \frac{r^2}{(1+r^2)^2}, \quad \psi_1(r) = \frac{1}{(1+r^2)^2} [r^4 + 4r^2 \log r - 1].$$

with the Wronskian :

$$W = \psi_1' \psi_0 - \psi_1 \psi_0' = \frac{rQ}{4} = \frac{2r}{(1+r^2)^2}. \quad (3.17)$$

Hence a solution to

$$L_0 m = -f$$

can be found by the method of variation of constants:

$$m = A\psi_0 + B\psi_1 \quad \text{with} \quad \begin{cases} A'\psi_0 + B'\psi_1 = 0, \\ A'\psi_0' + B'\psi_1' = f, \end{cases} \quad (3.18)$$

Hence, we obtain

$$B' = \frac{f\psi_0}{W} = \frac{r}{2}f, \quad A' = -\frac{f\psi_1}{W} = -\frac{r^4 + 4r^2 \log r - 1}{2r}f$$

and a solution is given by:

$$m(r) = -\frac{1}{2}\psi_0(r) \int_0^r \frac{\tau^4 + 4\tau^2 \log \tau - 1}{\tau} f(\tau) d\tau + \frac{1}{2}\psi_1(r) \int_0^r \tau f(\tau) d\tau. \quad (3.19)$$

Moreover, we compute

$$\frac{\psi_0'}{r} = \frac{2(1-r^2)}{(1+r^2)^3} = \frac{\Lambda Q}{8}, \quad \frac{\psi_1'}{r} = \frac{8(1+r^2 - (r^2-1)\log r)}{(1+r^2)^3}. \quad (3.20)$$

and then (3.18) yields

$$\begin{aligned} \frac{m'}{r} &= A \frac{\psi_0'}{r} + B \frac{\psi_1'}{r} \\ &= -\frac{1-r^2}{(1+r^2)^3} \int_0^r \frac{\tau^4 + 4\tau^2 \log \tau - 1}{\tau} f(\tau) d\tau + \frac{4(1+r^2 - (r^2-1)\log r)}{(1+r^2)^3} \int_0^r \tau f(\tau) d\tau. \end{aligned} \quad (3.21)$$

To conclude this part, we will need inverse an other operator, coming from the second equation :

$$L_1 d = d'' - \frac{d'}{r} = f \quad (3.22)$$

where f is given. $d_0(r) = r^2$ and $d_1(r) = 1$ are a basis of the homogeneous problem. Hence, a solution of (3.22) is given by

$$d = \frac{1}{2} \left[- \int_0^r f(\tau) \tau d\tau + r^2 \int_0^r \frac{f(\tau)}{\tau} d\tau \right] + cr^2, \quad c \in \mathbb{R}. \quad (3.23)$$

According to the definition of d , we have the constraint $d(0) = 0$. Hence, we consider only the solutions of the above form. In fact, $d_1(r)$ corresponds to the singular solution at the origin of $\Delta u = 0$.

We are now in position to determine the approximate solution :

Proposition 3.1 (Construction of the approximate profile). *Let $M > 0$ enough large. Then, there exists a small enough universal constant $b^*(M)$ such that the following holds. Let $b \in]0, b^*(M)[$, and B_0 et B_1 given by (1.14). There exists radial profiles T_1, T_2, S_1 et S_2 , such that*

$$\mathbf{Q}_b = \begin{vmatrix} Q_b(r) \\ P_b(r) \end{vmatrix} = \mathbf{Q} + b \begin{vmatrix} T_1(r) \\ S_1(r) \end{vmatrix} + b^2 \begin{vmatrix} T_2(b, r) \\ S_2(b, r) \end{vmatrix} = \mathbf{Q} + \mathbf{Y}_b = \mathbf{Q} + \begin{vmatrix} \alpha_b(r) \\ \gamma_b(r) \end{vmatrix} \quad (3.24)$$

is an approximate solution of (1.2) in the following sense. Let the error:

$$\mathbf{\Psi}_b = \begin{vmatrix} \Psi_b^{(1)} \\ \Psi_b^{(2)} \end{vmatrix} = \begin{vmatrix} \nabla \cdot (\nabla Q_b + Q_b \nabla P_b) \\ \Delta P_b - Q_b \end{vmatrix} - b\Lambda \begin{vmatrix} Q_b(r) \\ P_b(r) \end{vmatrix} + c_b b^2 \check{\mathbf{T}} \quad (3.25)$$

with c_b given by (3.50), and $\check{\mathbf{T}}$ such that

$$\begin{vmatrix} \check{T}^{(1)} \\ \nabla \check{T}^{(2)} \end{vmatrix} = \chi_{\frac{B_0}{4}} \begin{vmatrix} T_1 \\ \nabla S_1 \end{vmatrix} \quad (3.26)$$

Then there holds:

(i) Control of the tails: $\forall r \geq 0, \forall i \geq 0$:

$$|r^i \partial_r^i T_1| \lesssim \frac{r^2}{1+r^4}, \quad (3.27)$$

$$|r^i \partial_r^i \nabla S_1| \lesssim \frac{r}{1+r^2}, \quad (3.28)$$

and $\forall r \leq 2B_1, \forall i \geq 0$:

$$|r^i \partial_r^i T_2| \lesssim r^2 \mathbf{1}_{r \leq 1} + \frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^2 r^4 |\log b|} \mathbf{1}_{r \geq 6B_0}, \quad (3.29)$$

$$|b \partial_b r^i \partial_r^i T_2| \lesssim \frac{1}{|\log b|} \left[r^2 \mathbf{1}_{r \leq 1} + \frac{1 + |\log r|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^2 r^4} \mathbf{1}_{r \geq 6B_0} \right], \quad (3.30)$$

$$|r^i \partial_r^i \nabla S_2| \lesssim r \mathbf{1}_{r \leq 1} + r(1 + |\log r|) \mathbf{1}_{r \geq 1}, \quad (3.31)$$

$$|b \partial_b \nabla S_2| \lesssim \frac{1}{|\log b|^2} \{ r \mathbf{1}_{r \leq 1} + r(1 + |\log r|^2) \mathbf{1}_{r \geq 1} \}. \quad (3.32)$$

(ii) Control of the error in weighted norms: for $i \geq 0$,

$$\int_{r \leq 2B_1} |r^i \partial_r^i \Psi_b^{(1)}|^2 + \int_{r \leq 2B_1} \frac{|\mathcal{L}^{(1)}(\Psi_b^{(1)}, \Psi_b^{(2)})|^2}{Q} \lesssim \frac{b^5}{|\log b|^2}, \quad (3.33)$$

$$\int_{r \leq 2B_1} \frac{|\nabla \Psi_b^{(2)}|^2}{1+\tau^2} + \int_{r \leq 2B_1} |\mathcal{L}^{(2)}(\Psi_b^{(1)}, \Psi_b^{(2)})|^2 \lesssim b^5 |\log b|^2,$$

$$\int_{r \leq 2B_1} Q |\nabla \mathcal{M}^{(1)}(\Psi_b^{(1)}, \Psi_b^{(2)})|^2 \lesssim \frac{b^4}{|\log b|^2}. \quad (3.34)$$

$$\int_{r \leq 2B_1} |\nabla \Psi_b^{(2)}|^2 \lesssim b^4 |\log b|^6. \quad (3.35)$$

Proof. Step 1 : Computation of (Φ_b, Ω_b)

We recall that :

$$\begin{vmatrix} \Phi_b \\ \Omega_b \end{vmatrix} = \begin{vmatrix} m_b'' - \frac{m_b'}{r} + \frac{m_b' m_b}{r} + \frac{m_b' d_b}{r} - b r m_b' \\ d_b'' - \frac{d_b'}{r} - b r n_b' \end{vmatrix}$$

Injecting the expansions of m_b and d_b (3.14), and the definition of the operators L_0 and L_1 , we obtain

$$\begin{aligned} \begin{vmatrix} \Phi_b \\ \Omega_b \end{vmatrix} &= b \left\{ \mathbf{L} \begin{vmatrix} m_1 \\ d_1 \end{vmatrix} - \begin{vmatrix} rm'_0 \\ rn'_0 \end{vmatrix} \right\} + b^2 \left\{ \mathbf{L} \begin{vmatrix} m_2 \\ d_2 \end{vmatrix} - \begin{vmatrix} rm'_1 - \frac{m'_1 m_1}{r} - \frac{m'_1 d_1}{r} \\ rn'_1 \end{vmatrix} \right\} \\ &+ b^3 \begin{vmatrix} -rm'_2 + \frac{(m_1 m_2)'}{r} + \frac{m'_1 d_2 + m'_2 d_1}{r} \\ -rn'_2 \end{vmatrix} + b^4 \begin{vmatrix} \frac{m_2 m'_2}{r} + \frac{m'_2 d_2}{r} \\ 0 \end{vmatrix} \end{aligned} \quad (3.36)$$

where

$$\mathbf{L} \begin{vmatrix} m \\ d \end{vmatrix} = \begin{vmatrix} -L_0 m + \frac{m'_0}{r} d \\ L_1 d \end{vmatrix} \quad (3.37)$$

Step 1 : Level b

We look for m_1 and d_1 such that

$$\mathbf{L} \begin{vmatrix} m_1 \\ d_1 \end{vmatrix} = \begin{vmatrix} rm'_0 \\ rn'_0 \end{vmatrix}$$

First with (3.22), a solution of $L_1 d_1 = rn'_0$ is given for $c \in \mathbb{R}$ by:

$$\begin{aligned} d_{1,c}(r) &= \frac{1}{2} \left[- \int_0^r \frac{8\tau^3 d\tau}{(1+\tau^2)^2} + r^2 \int_0^r \frac{8\tau d\tau}{(1+\tau^2)^2} \right] + cr^2 \\ &= \frac{1}{2} \left[- \int_0^r \frac{8(\tau^3 + \tau) d\tau}{(1+\tau^2)^2} + (r^2 + 1) \int_0^r \frac{8\tau d\tau}{(1+\tau^2)^2} \right] + cr^2 \\ &= -2\log(1+r^2) + (2+c)r^2. \end{aligned}$$

Hence, we select as solution :

$$d_1(r) = d_{1,-2}(r) = -2\log(1+r^2). \quad (3.38)$$

We are in position to determine m_1 be the solution to :

$$L_0 m_1 = -rm'_0 \left(1 - \frac{d_1}{r^2} \right) \quad (3.39)$$

given by

$$\begin{aligned} m_1(r) &= -4\psi_0(r) \int_0^r \frac{\tau(\tau^4 + 4\tau^2 \log \tau - 1)}{(1+\tau^2)^2} \left(1 - \frac{d_1}{\tau^2} \right) d\tau \\ &+ 4\psi_1(r) \int_0^r \frac{\tau^3}{(1+\tau^2)^2} \left(1 - \frac{d_1}{\tau^2} \right) d\tau. \end{aligned}$$

Using the explicit formula :

$$\int_0^r \frac{\tau^3 d\tau}{(1+\tau^2)^2} = \frac{\log(1+r^2)}{2} + \frac{1}{2} \left(\frac{1}{1+r^2} - 1 \right), \quad (3.40)$$

we obtain :

$$m_1 = \begin{cases} O(r^4) & \text{at the origin} \\ 2\log(1+r^2) - 4 + O\left(\frac{|\log r|^2}{r^2}\right) = 4(\log r - 1) + O\left(\frac{|\log r|^2}{r^2}\right) & \text{as } r \rightarrow +\infty \end{cases} \quad (3.41)$$

Hence, using that $T_1 = \frac{m'_1}{r}$, there holds the behavior at the origin

$$T_1 = O(r^2) \quad (3.42)$$

and, for r large :

$$T_1(r) = \frac{4}{r^2} + O\left(\frac{|\log r|^2}{r^4}\right) \quad (3.43)$$

In particular, this yields the bound for $i \geq 0$:

$$|r^i \partial_r^i T_1| \lesssim \frac{r^2}{1+r^4}. \quad (3.44)$$

Now, by definition

$$n_1 = d_1 + m_1 \quad (3.45)$$

Hence, with (3.38) and (3.41):

$$n_1 = \begin{cases} O(r^4) & \text{at the origin} \\ -4 + O\left(\frac{|\log r|^2}{r^2}\right) & \text{as } r \rightarrow +\infty \end{cases} \quad (3.46)$$

and

$$|rn'_1| \lesssim r^4 \mathbf{1}_{r \leq 1} + \frac{1 + |\log r|^2}{r^2} \mathbf{1}_{r \geq 1}. \quad (3.47)$$

Thus, we obtain :

$$\nabla S_1 = \frac{n_1}{r} = \begin{cases} O(r) & \text{at the origin} \\ -\frac{4}{r} + O\left(\frac{|\log r|^2}{r^3}\right) & \text{as } r \rightarrow +\infty. \end{cases} \quad (3.48)$$

Step 2 : Construction of the radiation

In this step, we construct the term of radiation which allow us to reduce the growth of the profile T_2 , outside the parabolic zone ($y \geq B_0$). We will see that the choice of the radiation impose the law of b , and thus, the dynamics of the blow-up regime.

We let the radiation $\Sigma_{\mathbf{b}} = \begin{vmatrix} \Sigma_{1,b} \\ \nabla \Sigma_{2,b} \end{vmatrix}$ defined by $(m_{\Sigma_{\mathbf{b}}}, d_{\Sigma_{\mathbf{b}}})$ be the solution of

$$\mathbf{L} \begin{vmatrix} m_{\Sigma_{\mathbf{b}}} \\ d_{\Sigma_{\mathbf{b}}} \end{vmatrix} = c_b \chi_{\frac{B_0}{4}} \begin{vmatrix} rm'_0 \\ rn'_0 \end{vmatrix} + (1 - \chi_{3B_0}) \mathbf{L} \begin{vmatrix} \beta_{1,b} rm'_0 \\ \beta_{2,b} r^2 + \beta_{3,b} \end{vmatrix} \quad (3.49)$$

with c_b must be determined. We will find c_b such that:

$$c_b = \frac{2}{|\log b|} \left[1 + O\left(\frac{1}{|\log b|}\right) \right]. \quad (3.50)$$

Let

$$\beta_{2,b} = \int_0^{+\infty} \frac{\psi_0(\tau)}{\tau} (1 - \chi_{\frac{B_0}{4}}) d\tau = O(b), \quad (3.51)$$

$$\beta_{3,b} = \int_0^{+\infty} \tau \psi_0(\tau) \chi_{\frac{B_0}{4}} d\tau = O(|\log b|). \quad (3.52)$$

A solution of (3.49) is given by

$$d_{\Sigma_{\mathbf{b}}} = 4c_b \left[- \int_0^r \tau \psi_0(\tau) \chi_{\frac{B_0}{4}} d\tau + r^2 \int_0^r \frac{\psi_0(\tau)}{\tau} \chi_{\frac{B_0}{4}} d\tau - \frac{1}{2} r^2 + (1 - \chi_{3B_0}) \{ \beta_{2,b} r^2 + \beta_{3,b} \} \right] \quad (3.53)$$

and

$$\begin{aligned} m_{\Sigma_{\mathbf{b}}} &= -4c_b \psi_0(r) \int_0^r \frac{\tau(\tau^4 + 4\tau^2 \log \tau - 1)}{(1 + \tau^2)^2} \left(\chi_{\frac{B_0}{4}} - \frac{d_{\Sigma_{\mathbf{b}}}}{\tau^2} \right) d\tau \\ &+ 4c_b \psi_1(r) \int_0^r \frac{\tau^3}{(1 + \tau^2)^2} \left(\chi_{\frac{B_0}{4}} - \frac{d_{\Sigma_{\mathbf{b}}}}{\tau^2} \right) d\tau + \beta_{1,b} (1 - \chi_{3B_0}) \psi_0(r). \end{aligned} \quad (3.54)$$

with

$$\beta_{1,b} = 4c_b \int_0^{+\infty} \frac{\tau(\tau^4 + 4\tau^2 \log \tau - 1)}{(1 + \tau^2)^2} \left(\chi_{\frac{B_0}{4}} - \frac{d_{\Sigma_{\mathbf{b}}}}{\tau^2} \right) d\tau = O\left(\frac{1}{b|\log b|}\right) \quad (3.55)$$

Look for c_b such that:

$$c_b \int_0^{+\infty} \frac{\tau^3}{(1+\tau^2)^2} \left(\chi_{\frac{B_0}{4}} - \frac{d_{\Sigma_{\mathbf{b}}}}{\tau^2} \right) d\tau = 1. \quad (3.56)$$

Let

$$\begin{aligned} c_{1,b} &= \int_0^{+\infty} \frac{\tau^3}{(1+\tau^2)^2} \chi_{\frac{B_0}{4}} = \frac{|\log b|}{2} + O(1), \\ c_{2,b} &= \int_0^{+\infty} \frac{\tau^3}{(1+\tau^2)^2} \frac{d_{\Sigma_{\mathbf{b}}}}{c_b \tau^2} d\tau = O(1). \end{aligned}$$

Then, the following c_b satisfy the constraint (3.56):

$$c_b = \frac{c_{1,b} - \sqrt{c_{1,b}^2 - 4c_{2,b}}}{2c_{2,b}} = \frac{2}{|\log b|} + O\left(\frac{1}{|\log b|^2}\right). \quad (3.57)$$

Observe that by definition:

$$\left| \begin{array}{l} m_{\Sigma_{\mathbf{b}}} \\ d_{\Sigma_{\mathbf{b}}} \end{array} \right| = c_b \left| \begin{array}{l} m_1 \\ d_1 \end{array} \right| \quad \text{for } r \leq \frac{B_0}{4} \quad (3.58)$$

and

$$\left| \begin{array}{l} m_{\Sigma_{\mathbf{b}}} \\ d_{\Sigma_{\mathbf{b}}} \end{array} \right| = \left| \begin{array}{l} 4\psi_1 \\ 0 \end{array} \right| \quad \text{for } r \geq 6B_0. \quad (3.59)$$

Now, we can estimate for $\frac{B_0}{4} \leq r \leq 6B_0$

$$\left| \begin{array}{l} m_{\Sigma_{\mathbf{b}}} \\ d_{\Sigma_{\mathbf{b}}} \end{array} \right| = \left| \begin{array}{l} 4 + O\left(\frac{1}{|\log b|^2}\right) \\ O(1) \end{array} \right| \quad (3.60)$$

Now, we are in position to estimate $\Sigma_{\mathbf{b}}$ and its derivatives. First, by construction, we have

$$\left| \begin{array}{l} \Sigma_{1,b} \\ \nabla \Sigma_{2,b} \end{array} \right| = c_b \left| \begin{array}{l} T_1 \\ \nabla S_1 \end{array} \right| \quad \text{for } r \leq \frac{B_0}{4}. \quad (3.61)$$

Moreover, we recall that :

$$\Sigma_{1,b}(r) = \frac{m'_{\Sigma_{\mathbf{b}}}(r)}{r} \quad \text{and} \quad \nabla \Sigma_{2,b} = \frac{d_{\Sigma_{\mathbf{b}}}(r) + m_{\Sigma_{\mathbf{b}}}(r)}{r}$$

Hence, with (3.59) and the above formula, we obtain for $r \geq 6B_0$:

$$\left| \begin{array}{l} \Sigma_{1,b} \\ \nabla \Sigma_{2,b} \end{array} \right| = \left| \begin{array}{l} 4\frac{\psi'_1}{r} \\ \frac{4\phi_1}{r} \end{array} \right| = \left| \begin{array}{l} O\left(\frac{\log r}{r^4}\right) \\ \frac{4}{r} + O\left(\frac{\log r}{r^3}\right) \end{array} \right| \quad (3.62)$$

To obtain a precise bound in the transition zone, we use the improved formula (3.21):

$$\begin{aligned} \frac{m'_{\Sigma_{\mathbf{b}}}}{r} &= -4c_b \frac{\psi'_0(r)}{r} \int_0^r \frac{\tau(\tau^4 + 4\tau^2 \log \tau - 1)}{(1+\tau^2)^2} \left(\chi_{\frac{B_0}{4}} - \frac{d_{\Sigma_{\mathbf{b}}}}{\tau^2} \right) d\tau - \beta_{1,b} \chi'_{3B_0} \frac{\psi_0}{r} \\ &+ 4c_b \frac{\psi'_1(r)}{r} \int_0^r \frac{\tau^3}{(1+\tau^2)^2} \left(\chi_{\frac{B_0}{4}} - \frac{d_{\Sigma_{\mathbf{b}}}}{\tau^2} \right) d\tau + \beta_{1,b} (1 - \chi_{3B_0}) \frac{\psi'_0(r)}{r} \end{aligned} \quad (3.63)$$

Hence, for $\frac{B_0}{4} \leq y \leq 6B_0$, (3.63) together with (3.60) give :

$$|\Sigma_{1,b}(r)| \lesssim \frac{1}{|\log b| r^2} \quad \text{and} \quad |\nabla \Sigma_{2,b}(r)| \lesssim \frac{1}{r} \quad (3.64)$$

The above estimation with (3.61) and (3.62) imply the following rough bounds, for $i \geq 0$ and $r \leq 2B_1$:

$$|r^i \partial_r^i \Sigma_{1,b}(r)| \lesssim \frac{1}{|\log b|} \left[r^2 \mathbf{1}_{r \leq 1} + \frac{1}{r^2} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{br^4} \mathbf{1}_{r \geq 6B_0} \right] \quad (3.65)$$

$$|r^i \partial_r^i \nabla \Sigma_{2,b}(r)| \lesssim \frac{1}{|\log b|} \left[r \mathbf{1}_{r \leq 1} + \frac{1}{r} \mathbf{1}_{1 \leq r \leq \frac{B_0}{4}} + \frac{|\log b|}{r} \mathbf{1}_{r \geq \frac{B_0}{4}} \right] \quad (3.66)$$

Before determining m_2 and d_2 , compute the b dependence of $\Sigma_{\mathbf{b}}$. First we have :

$$\frac{\partial c_b}{\partial b} = O\left(\frac{1}{b|\log b|^2}\right), \quad \frac{\partial \beta_{1,b}}{\partial b} = O\left(\frac{1}{b^2|\log b|}\right), \quad \frac{\partial \beta_{2,b}}{\partial b} = O(1), \quad \frac{\partial \beta_{3,b}}{\partial b} = O\left(\frac{1}{b}\right).$$

By definition,

$$\frac{\partial}{\partial b} \left| \begin{matrix} m_{\Sigma_{\mathbf{b}}} \\ d_{\Sigma_{\mathbf{b}}} \end{matrix} \right| = \partial_b c_b \left| \begin{matrix} m_1 \\ d_1 \end{matrix} \right| = O\left(\frac{1}{b|\log b|^2}\right) \left| \begin{matrix} m_1 \\ d_1 \end{matrix} \right| \quad \text{for } r \leq \frac{B_0}{4}. \quad (3.67)$$

Now

$$\frac{\partial}{\partial b} \left| \begin{matrix} m_{\Sigma_{\mathbf{b}}} \\ d_{\Sigma_{\mathbf{b}}} \end{matrix} \right| = \left| \begin{matrix} 0 \\ 0 \end{matrix} \right| \quad \text{for } r \geq 6B_0. \quad (3.68)$$

To conclude, in the transition zone $\frac{B_0}{4} \leq r \leq 6B_0$

$$\begin{aligned} \left| \frac{\partial d_{\Sigma_{\mathbf{b}}}}{\partial b} \right| &\lesssim \frac{1}{b|\log b|} + |c_b| \int_0^r \frac{1}{\tau} \left| \partial_b \chi_{\frac{B_0}{4}} \right| d\tau + |c_b| r^2 \int_0^r \frac{1}{\tau^3} \left| \partial_b \chi_{\frac{B_0}{4}} \right| d\tau \\ &\quad + |c_b| \left| \frac{\partial \{ \chi_{3B_0}(\beta_{2,b} r^2 + \beta_{1,b}) \}}{\partial b} \right| \\ &\lesssim \frac{1}{b}. \end{aligned}$$

Using the same way, we prove that in the transition zone,

$$\left| \frac{\partial m_{\Sigma_{\mathbf{b}}}}{\partial b} \right| \lesssim \frac{1}{b|\log b|}. \quad (3.69)$$

This yields the bound for $r \leq 2B_1$:

$$\left| \frac{\partial m_{\Sigma_{\mathbf{b}}}(r)}{\partial b} \right| \lesssim \frac{1}{b|\log b|^2} [r^4 \mathbf{1}_{r \leq 1} + (1 + \log r) \mathbf{1}_{1 \leq r \leq 6B_0}] \quad (3.70)$$

$$\left| \frac{\partial d_{\Sigma_{\mathbf{b}}}(r)}{\partial b} \right| \lesssim \frac{1}{b|\log b|^2} [r^2 \mathbf{1}_{r \leq 1} + (1 + \log r)^2 \mathbf{1}_{1 \leq r \leq 6B_0}] \quad (3.71)$$

and the following bound for $i \geq 1$ and $r \leq 2B_1$ using the explicit formula of ψ'_1 , m'_1 and d'_1 :

$$\left| r^i \partial_r^i \frac{\partial m_{\Sigma_{\mathbf{b}}}}{\partial b}(r) \right| \lesssim \frac{1}{b|\log b|^2} [r^4 \mathbf{1}_{r \leq 1} + \mathbf{1}_{1 \leq r \leq 6B_0}] \quad (3.72)$$

$$\left| r^i \partial_r^i \frac{\partial d_{\Sigma_{\mathbf{b}}}}{\partial b}(r) \right| \lesssim \frac{1}{b|\log b|^2} [r^2 \mathbf{1}_{r \leq 1} + (1 + \log r)^3 \mathbf{1}_{1 \leq r \leq 6B_0}] \quad (3.73)$$

Step 3 : Level b^2

We are now in position to determine (m_2, d_2) be solution to

$$\mathbf{L} \begin{vmatrix} m_2 \\ d_2 \end{vmatrix} = \begin{vmatrix} r m'_1 - \frac{m'_1 m_1}{r} - \frac{m'_1 d_1}{r} \\ r n'_1 \end{vmatrix} - \begin{vmatrix} m_{\Sigma_{\mathbf{b}}} \\ d_{\Sigma_{\mathbf{b}}} \end{vmatrix} = \begin{vmatrix} m_{\Sigma_2} \\ d_{\Sigma_2} \end{vmatrix} \quad (3.74)$$

Hence, from (3.23) , d_2 is given by:

$$d_2 = \frac{1}{2} \left[- \int_0^r d_{\Sigma_2}(\tau) \tau d\tau + r^2 \int_0^r \frac{d_{\Sigma_2}(\tau)}{\tau} d\tau \right] \quad (3.75)$$

With the estimations (3.47), (3.58) and (3.60), we obtain the rough bound:

$$|d_2(r)| \lesssim r^2 \mathbf{1}_{r \leq 1} + r^2 (1 + \log r) \mathbf{1}_{r \geq 1}. \quad (3.76)$$

Moreover, we have the following bound using (3.71) and (??):

$$\begin{aligned} |b\partial_b d_2(r)| &\lesssim \frac{1}{2} \left[\int_0^r |b\partial_b d_{\Sigma_b}(\tau)| \tau d\tau + r^2 \int_0^r \left| \frac{b\partial_b d_{\Sigma_b}(\tau)}{\tau} \right| d\tau \right] + |b\partial_b \delta_b| r^2 \\ &\lesssim \frac{r^2}{|\log b|^2} \mathbf{1}_{r \leq 1} + r^2 \frac{1 + |\log r|^2}{|\log b|^2} \mathbf{1}_{1 \leq r \leq 6B_0} + r^2 \mathbf{1}_{r \geq 6B_0}. \end{aligned} \quad (3.77)$$

Now m_2 is given by:

$$\begin{aligned} m_2(r) &= -\frac{1}{2} \psi_0(r) \int_0^r \frac{\tau^4 + 4\tau^2 \log \tau - 1}{\tau} [Q(\tau) d_2(\tau) - m_{\Sigma_2}(\tau)] d\tau \\ &\quad + \frac{1}{2} \psi_1(r) \int_0^r \tau [Q(\tau) d_2(\tau) - m_{\Sigma_2}(\tau)] d\tau. \end{aligned}$$

Let

$$g_b(\tau) = Q(\tau) d_2(\tau) - m_{\Sigma_2}(\tau). \quad (3.78)$$

Using the definition of m_{Σ_2} and the above estimations :

$$|g_b(r)| \lesssim r^2 \mathbf{1}_{r \leq 1} + \frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{|\log r|^2}{1 + r^2} \mathbf{1}_{r \geq 6B_0}.$$

and

$$|b\partial_b g_b(r)| \lesssim \frac{1}{|\log b|^2} [r^4 \mathbf{1}_{r \leq 1} + (1 + \log r) \mathbf{1}_{1 \leq r \leq 6B_0}] + \frac{1}{r^2} \mathbf{1}_{r \geq 6B_0}. \quad (3.79)$$

Hence, near the origin, we have

$$m_2 = O(r^4).$$

For $1 \leq r \leq 6B_0$,

$$\begin{aligned} |m_2(r)| &\lesssim \frac{1}{1 + r^2} \int_0^r \tau^3 \frac{1 + |\log(\tau\sqrt{b})|}{|\log b|} d\tau + \int_0^r \tau \frac{1 + |\log(\tau\sqrt{b})|}{|\log b|} d\tau \\ &\lesssim r^2 \frac{1 + |\log(r\sqrt{b})|}{|\log b|}. \end{aligned}$$

For $r \geq 6B_0$,

$$\begin{aligned} |m_2(r)| &\lesssim \frac{1}{b|\log b|} + \frac{1}{r^2} \int_{6B_0}^r \tau^3 \frac{|\log \tau|^2}{1 + \tau^2} d\tau + \int_{6B_0}^r \tau \frac{|\log \tau|^2}{1 + \tau^2} d\tau \\ &\lesssim \frac{1}{b|\log b|} + |\log r|^3. \end{aligned}$$

Now, outside the parabolic zone, for $6B_0 \leq r \leq 2B_1$,

$$\begin{aligned}
\left| \frac{\partial_r m_2}{r} \right| &= \left| \frac{1}{2} \frac{\psi'_0(r)}{r} \int_0^r \frac{\tau^4 + 4\tau^2 \log \tau - 1}{\tau} g_b(\tau) d\tau - \frac{1}{2} \frac{\psi'_1(r)}{r} \int_0^r \tau g_b(\tau) d\tau \right| \\
&\lesssim \frac{1}{r^4} \int_0^r \frac{\tau^4 + 4\tau^2 \log \tau - 1}{\tau} \left[r^2 \mathbf{1}_{r \leq 1} + \frac{1 + |\log(\tau \sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq \tau \leq 6B_0} + \frac{|\log \tau|^2}{1 + \tau^2} \mathbf{1}_{\tau \geq 6B_0} \right] d\tau \\
&+ \frac{|\log r|}{r^4} \int_0^r \tau \left[r^2 \mathbf{1}_{r \leq 1} + \frac{1 + |\log(\tau \sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq \tau \leq 6B_0} + \frac{|\log \tau|^2}{1 + \tau^2} \mathbf{1}_{\tau \geq 6B_0} \right] d\tau \\
&\lesssim \frac{1}{r^4} \left[\frac{1}{b^2 |\log b|} + r^2 (\log r)^2 \right] + \frac{\log r}{r^4} \left[\frac{1}{b |\log b|} + |\log r|^3 \right] \\
&\lesssim \frac{1}{r^4 b^2 |\log b|}. \tag{3.80}
\end{aligned}$$

The collection of above bounds yields the control: $\forall r \leq 2B_1$

$$|m_2| \lesssim r^4 \mathbf{1}_{r \leq 1} + r^2 \frac{1 + |\log(r \sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b |\log b|} \mathbf{1}_{r \geq 6B_0}, \tag{3.81}$$

$$|r^i \partial_r^i m_2| \lesssim r^4 \mathbf{1}_{r \leq 1} + r^2 \frac{1 + |\log(r \sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{r^2 b^2 |\log b|} \mathbf{1}_{r \geq 6B_0}, \quad i \geq 1. \tag{3.82}$$

The b dependance is estimated using (3.79):

$$\begin{aligned}
|b \partial_b m_2| &\lesssim \frac{r^2}{1 + r^4} \int_0^r \frac{1}{|\log b|^2} [\tau^7 \mathbf{1}_{\tau \leq 1} + \tau^3 (1 + \log \tau) \mathbf{1}_{1 \leq \tau \leq 6B_0}] + \tau \mathbf{1}_{\tau \geq 6B_0} d\tau \\
&+ \int_0^r \frac{1}{|\log b|^2} [\tau^5 \mathbf{1}_{\tau \leq 1} + \tau (1 + \log \tau) \mathbf{1}_{1 \leq \tau \leq 6B_0}] + \frac{1}{\tau} \mathbf{1}_{\tau \geq 6B_0} d\tau \\
&\lesssim \frac{1}{|\log b|^2} \left[r^6 \mathbf{1}_{r \leq 1} + r^2 (1 + \log r) \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{\log b}{b} \mathbf{1}_{r \geq 6B_0} \right] \tag{3.83}
\end{aligned}$$

and for higher derivatives:

$$\begin{aligned}
\left| \frac{b \partial_b \partial_r m_2}{r} \right| &\lesssim \frac{1}{1 + r^4} \int_0^r \frac{1}{|\log b|^2} [\tau^7 \mathbf{1}_{\tau \leq 1} + \tau^3 (1 + \log \tau) \mathbf{1}_{1 \leq \tau \leq 6B_0}] + \tau \mathbf{1}_{\tau \geq 6B_0} d\tau \\
&+ \frac{1 + |\log r|}{1 + r^4} \int_0^r \frac{1}{|\log b|^2} [\tau^5 \mathbf{1}_{\tau \leq 1} + \tau (1 + \log \tau) \mathbf{1}_{1 \leq \tau \leq 6B_0}] + \frac{1}{\tau} \mathbf{1}_{\tau \geq 6B_0} \\
&\lesssim \frac{1}{|\log b|^2} [r^4 \mathbf{1}_{r \leq 1} + (1 + \log r) \mathbf{1}_{1 \leq r \leq 6B_0}] + \frac{1}{b^2 r^4 |\log b|} \mathbf{1}_{r \geq 6B_0} \tag{3.84}
\end{aligned}$$

and hence the bounds for $r \leq 2B_1$:

$$|b \partial_b m_2| \lesssim \frac{1}{|\log b|} \left[r^4 \mathbf{1}_{r \leq 1} + r^2 \frac{1 + |\log r|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b} \mathbf{1}_{r \geq 6B_0} \right], \tag{3.85}$$

and for $i \geq 1$:

$$|b \partial_b r^i \partial_r^i m_2| \lesssim \frac{1}{|\log b|} \left[r^4 \mathbf{1}_{r \leq 1} + r^2 \frac{1 + |\log r|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{r^2 b^2} \mathbf{1}_{r \geq 6B_0} \right]. \tag{3.86}$$

This yields the estimate on $T_2 = \frac{m'_2}{r}$: for $i \geq 0$, $r \leq 2B_1$:

$$\begin{aligned}
|r^i \partial_r^i T_2| &\lesssim r^2 \mathbf{1}_{r \leq 1} + \frac{1 + |\log(r \sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^2 r^4 |\log b|} \mathbf{1}_{r \geq 6B_0}, \\
|b \partial_b r^i \partial_r^i T_2| &\lesssim \frac{1}{|\log b|} \left[r^2 \mathbf{1}_{r \leq 1} + \frac{1 + |\log r|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^2 r^4} \mathbf{1}_{r \geq 6B_0} \right].
\end{aligned}$$

Using the fact that $\nabla S_2 = \frac{d_2(r)+m_2(r)}{r}$, we have the following rough bounds:

$$|r^i \partial_r^i \nabla S_2| \lesssim r \mathbf{1}_{r \leq 1} + r(1 + |\log r|) \mathbf{1}_{r \geq 1} \quad (3.87)$$

and

$$|r^i \partial_r^i b \partial_b \nabla S_2| \lesssim \frac{1}{|\log b|^2} \{r \mathbf{1}_{r \leq 1} + r(1 + |\log r|^2) \mathbf{1}_{r \geq 1}\} \quad (3.88)$$

Step 4 : Estimate on the error:

According to the construction of the profiles T_i and S_i , $\left| \begin{array}{c} \Phi_b \\ \Omega_b \end{array} \right|$ satisfies the following equation :

$$\left| \begin{array}{c} \Phi_b \\ \Omega_b \end{array} \right| = -b^2 \left| \begin{array}{c} m_{\Sigma_b} \\ d_{\Sigma_b} \end{array} \right| + \left| \begin{array}{c} R^{(1)}(r) \\ R^{(2)}(r) \end{array} \right| \quad (3.89)$$

where

$$\left| \begin{array}{c} R^{(1)}(r) \\ R^{(2)}(r) \end{array} \right| = b^3 \left| \begin{array}{c} -rm'_2 + \frac{(m_1 m_2)'}{r} + \frac{m'_1 d_2 + m'_2 d_1}{r} \\ -rn'_2 \end{array} \right| + b^4 \left| \begin{array}{c} \frac{m_2 m'_2}{r} + \frac{m'_2 d_2}{r} \\ 0 \end{array} \right|. \quad (3.90)$$

Hence, from the definition (3.25) of the error, we obtain :

$$\left| \begin{array}{c} \Psi_b^{(1)} \\ \nabla \Psi_b^{(2)} \end{array} \right| = \frac{1}{r} \left| \begin{array}{c} \Phi'_b \\ \Omega_b \end{array} \right| + c_b b^2 \chi_{\frac{B_0}{4}} \left| \begin{array}{c} T_1 \\ \nabla S_1 \end{array} \right| = \frac{1}{r} \left| \begin{array}{c} (R^{(1)}(r))' \\ R^{(2)}(r) \end{array} \right| + b^2 \left| \begin{array}{c} c_b T_1 \chi_{\frac{B_0}{4}} - \Sigma_{1,b} \\ c_b \nabla S_1 \chi_{\frac{B_0}{4}} - \nabla \Sigma_{2,b} \end{array} \right|. \quad (3.91)$$

Using (3.38), (3.41), (3.76), (3.81) and (3.82), we prove the following bound for $i \geq 0$, remarking that the worst term is $-rm'_2$:

$$|r^i \partial_r^i R^{(1)}(r)| \lesssim b^3 \left[r^4 \mathbf{1}_{r \leq 1} + r^2 \frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{r^2 b^2 |\log b|} \mathbf{1}_{r \geq 6B_0} \right]. \quad (3.92)$$

Now, the bound (3.87) gives :

$$|r^i \partial_r^i R^{(2)}(r)| \lesssim b^3 \{r^2 \mathbf{1}_{r \leq 1} + r^2(1 + |\log r|) \mathbf{1}_{r \geq 1}\}. \quad (3.93)$$

According to the conception of the radiation Σ_b , and the estimations (3.65) and (3.66), this yields the bounds for $r \leq 2B_1$ and $i \geq 0$:

$$\begin{aligned} |r^i \partial_r^i \Psi_b^{(1)}| &\lesssim b^3 \left[r^2 \mathbf{1}_{r \leq 1} + \frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{r^4 b^2 |\log b|} \mathbf{1}_{r \geq 6B_0} \right] \\ &+ \frac{b^2}{|\log b|} \left[\frac{1}{r^2} \mathbf{1}_{\frac{B_0}{4} \leq r \leq 6B_0} + \frac{1}{b r^4} \mathbf{1}_{r \geq 6B_0} \right] \end{aligned} \quad (3.94)$$

$$\begin{aligned} |r^i \partial_r^i \nabla \Psi_b^{(2)}| &\lesssim b^3 [r \mathbf{1}_{r \leq 1} + r(1 + |\log r|) \mathbf{1}_{r \geq 1}] + \frac{b^2}{r} \mathbf{1}_{r \geq \frac{B_0}{4}} \\ &\lesssim b^3 [r \mathbf{1}_{r \leq 1} + r(1 + |\log r|) \mathbf{1}_{r \geq 1}] \end{aligned} \quad (3.95)$$

We therefore estimate:

$$\begin{aligned} \int_{r \leq 2B_1} |\Psi_b^{(1)}|^2 &\lesssim b^6 \int_{r \leq 2B_1} \left[r^4 \mathbf{1}_{r \leq 1} + \left(\frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} \right)^2 + \frac{1}{r^8 b^4 |\log b|^2} \mathbf{1}_{r \geq 6B_0} \right] \\ &+ \frac{b^4}{|\log b|^2} \int_{r \leq 2B_1} \left[\frac{1}{r^4} \mathbf{1}_{\frac{B_0}{4} \leq r \leq 6B_0} + \frac{1}{b^2 r^8} \mathbf{1}_{r \geq 6B_0} \right] \\ &\lesssim \frac{b^5}{|\log b|^2} \end{aligned}$$

and similarly for higher derivatives:

$$\int_{r \leq 2B_1} |r^i \partial_r^i \Psi_b^{(1)}|^2 \lesssim \frac{b^5}{|\log b|^2}.$$

Moreover, using the explicit formula of $\mathcal{L}^{(1)}$, and the bounds (3.94) and (3.95), we obtain :

$$\begin{aligned} \int_{r \leq 2B_1} \frac{|\mathcal{L}^{(1)}(\Psi_b^{(1)}, \Psi_b^{(2)})|^2}{Q} &\lesssim \sum_{i=0}^2 \int_{r \leq 2B_1} (1+r^{2i}) |\partial_r^i \Psi_b^{(1)}|^2 + \sum_{i=0}^1 \int_{r \leq 2B_1} \frac{|\partial_r^i \nabla \Psi_b^{(2)}|^2}{1+r^{6-2i}} \\ &\lesssim \frac{b^5}{|\log b|^2}. \end{aligned}$$

Now, with (3.95) :

$$\begin{aligned} \int_{r \leq 2B_1} \frac{|\nabla \Psi_b^{(2)}|^2}{1+r^2} + \int_{r \leq 2B_1} |\Delta \Psi_b^{(2)}|^2 &\lesssim b^6 \int_{r \leq 2B_1} [r^2 \mathbf{1}_{r \leq 1} + (1+\log r)^2 \mathbf{1}_{r \geq 1}] \\ &\lesssim b^5 |\log b|^2. \end{aligned} \quad (3.96)$$

Moreover,

$$\int_{r \leq 2B_1} |\nabla \Psi_b^{(2)}|^2 \lesssim b^6 \int_{r \leq 2B_1} [r^2 \mathbf{1}_{r \leq 1} + r^2 (1+\log r)^2 \mathbf{1}_{r \geq 1}] \lesssim b^4 |\log b|^6. \quad (3.97)$$

Finally, we estimate the following norm:

$$\begin{aligned} &\int_{r \leq 2B_1} Q |\nabla \mathcal{M}^{(1)}(\Psi_b^{(1)}, \Psi_b^{(2)})|^2 \\ &\lesssim \int_{r \leq 2B_1} (1+r^2) [|r \partial_r \Psi_b^{(1)}|^2 + |\Psi_b^{(1)}|^2] + \frac{|\nabla \Psi_b^{(2)}|^2}{1+r^4} \\ &\lesssim b^6 \int_{r \leq 2B_1} \left[1 + \frac{(1+|\log(r\sqrt{b})|^2)(1+r^2)}{|\log b|^2} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{r^6 b^4 |\log b|} \mathbf{1}_{r \geq 6B_0} \right] \\ &+ \frac{b^4}{|\log b|^2} \int_{r \leq 2B_1} \left[\frac{1}{r^2} \mathbf{1}_{\frac{B_0}{4} \leq r \leq 6B_0} + \frac{1}{b^2 r^6} \mathbf{1}_{r \geq 6B_0} \right] + \frac{b^5}{|\log b|^2} \\ &\lesssim \frac{b^4}{|\log b|^2}. \end{aligned}$$

This concludes the proof of the Proposition 3.1. \square

3.3. Localization. We can see that, outside the parabolic zone $r \geq 2B_1$, the profiles T_i and S_i have a pathological growth. Therefore, we must localize this profiles, as described in the following Proposition.

Proposition 3.2 (Localization). *Given a small parameter*

$$0 < b \ll 1. \quad (3.98)$$

Let the localized profiles

$$\tilde{\mathbf{Q}}_b = \begin{vmatrix} \tilde{Q}_b(r) \\ \tilde{P}_b(r) \end{vmatrix} = \mathbf{Q} + b \begin{vmatrix} \tilde{T}_1(r) \\ \tilde{S}_1(r) \end{vmatrix} + b^2 \begin{vmatrix} \tilde{T}_2(b, r) \\ \tilde{S}_2(b, r) \end{vmatrix} = \mathbf{Q} + \tilde{\mathbf{Y}}_b = \mathbf{Q} + \begin{vmatrix} \tilde{\alpha}_b(r) \\ \tilde{\gamma}_b(r) \end{vmatrix} \quad (3.99)$$

with

$$\tilde{T}_i = \chi_{B_1} T_i \text{ for } i = 1, 2. \quad (3.100)$$

and

$$\tilde{S}_i(r) = \int_0^r \chi_{B_1} \nabla S_i dr \quad \text{for } i = 1, 2. \quad (3.101)$$

Let the error:

$$\tilde{\Psi}_b = \begin{vmatrix} \tilde{\Psi}_b^{(1)} \\ \tilde{\Psi}_b^{(2)} \end{vmatrix} = \begin{vmatrix} \nabla \cdot (\nabla \tilde{Q}_b + \tilde{Q}_b \nabla \tilde{P}_b) \\ \Delta \tilde{P}_b - \tilde{Q}_b \end{vmatrix} - b\Lambda \begin{vmatrix} \tilde{Q}_b(r) \\ \tilde{P}_b(r) \end{vmatrix} + c_b b^2 \check{\mathbf{T}} \quad (3.102)$$

with c_b given by (3.50), and $\check{\mathbf{T}}$ satisfied the condition (3.26) then there holds:

(i) Control of the tails: $\forall r \geq 0, \forall i \geq 0$:

$$|r^i \partial_r^i \tilde{T}_1| \lesssim \frac{r^2}{1+r^4}, \quad (3.103)$$

$$|b \partial_b r^i \partial_r^i \tilde{T}_1| \lesssim \frac{1}{r^2} \mathbf{1}_{B_1 \leq r \leq 2B_1}, \quad (3.104)$$

$$|r^i \partial_r^i \nabla \tilde{S}_1| \lesssim \frac{r}{1+r^2}, \quad (3.105)$$

$$|b \partial_b r^i \partial_r^i \tilde{S}_1| \lesssim \frac{1}{r} \mathbf{1}_{B_1 \leq r \leq 2B_1}, \quad (3.106)$$

and $\forall r \leq 2B_1, \forall i \geq 0$:

$$|r^i \partial_r^i \tilde{T}_2| \lesssim r^4 \mathbf{1}_{r \leq 1} + \frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^2 r^4 |\log b|} \mathbf{1}_{r \geq 6B_0} \quad (3.107)$$

$$|b \partial_b r^i \partial_r^i \tilde{T}_2| \lesssim \frac{1}{|\log b|} \left[r^4 \mathbf{1}_{r \leq 1} + \frac{|\log r|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^2 r^4} \mathbf{1}_{r \geq 6B_0} \right]. \quad (3.108)$$

$$|r^i \partial_r^i \nabla \tilde{S}_2| \lesssim r \mathbf{1}_{r \leq 1} + r(1 + \log r) \mathbf{1}_{r \geq 1}, \quad (3.109)$$

$$|b \partial_b r^i \partial_r^i \nabla \tilde{S}_2| \lesssim \frac{1}{|\log b|^2} [r \mathbf{1}_{r \leq 1} + r(1 + \log r)^2 \mathbf{1}_{r \geq 1}]. \quad (3.110)$$

(ii) Control of the error in weighted norms: for $i \geq 0$,

$$\int |r^i \partial_r^i \tilde{\Psi}_b^{(1)}|^2 + \int \frac{|\mathcal{L}^{(1)}(\tilde{\Psi}_b^{(1)}, \tilde{\Psi}_b^{(2)})|^2}{Q} \lesssim \frac{b^5}{|\log b|^2}, \quad (3.111)$$

$$\int \frac{|\nabla \tilde{\Psi}_b^{(2)}|^2}{1+\tau^2} + \int |\mathcal{L}^{(2)}(\tilde{\Psi}_b^{(1)}, \tilde{\Psi}_b^{(2)})|^2 \lesssim b^5 |\log b|^4, \quad (3.112)$$

$$\int Q |\nabla \mathcal{M}^{(1)}(\tilde{\Psi}_b^{(1)}, \tilde{\Psi}_b^{(2)})|^2 \lesssim \frac{b^4}{|\log b|^2}, \quad (3.113)$$

$$\int |\nabla \tilde{\Psi}_b^{(2)}|^2 \lesssim b^4 |\log b|^6. \quad (3.114)$$

(iii) Degenerate flux: Let $\frac{B_0}{20} \leq B \leq 20B_0$ and $\Phi_{0,B} = \begin{vmatrix} r^2 \chi_B \\ -4 \int_0^r \frac{\log(1+\tau^2)}{\tau} \chi_B d\tau \end{vmatrix}$, then

$$\left| \langle \mathcal{L} \tilde{\Psi}_b, \Phi_{0,B} \rangle \right| \lesssim \frac{b^2}{|\log b|}. \quad (3.115)$$

Remark 3.3. We have fixed $\tilde{S}_i(0) = 0$. In fact, it isn't a necessity for our analysis, but there is an advantage. We have the equality $\phi_{\Delta \tilde{P}_b} = \tilde{P}_b$.

Proof. Step 1 Terms induced by the localization :

From (3.100) and (3.101), we have :

$$|\partial_r \tilde{T}_i| = \left| \chi_{B_1} \partial_r T_i + \frac{1}{B_1} \chi' \left(\frac{y}{B_1} \right) T_i \right| \lesssim |\partial_r T_i| \mathbf{1}_{r \leq 2B_1} + \left| \frac{T_i}{y} \right| \mathbf{1}_{B_1 \leq r \leq 2B_1},$$

$$|\partial_r \nabla \tilde{S}_i| = \left| \chi_{B_1} \partial_r \nabla S_i + \frac{1}{B_1} \chi' \left(\frac{y}{B_1} \right) \nabla S_i \right| \lesssim |\partial_r \nabla S_i| \mathbf{1}_{r \leq 2B_1} + \left| \frac{\nabla S_i}{y} \right| \mathbf{1}_{B_1 \leq r \leq 2B_1},$$

and

$$|b \partial_b \tilde{T}_i| = \left| b \chi_{B_1} \partial_b T_i - b \frac{\partial_b B_1}{B_1} (y \chi') \left(\frac{y}{B_1} \right) T_i \right| \lesssim |b \partial_b T_i| \mathbf{1}_{r \leq 2B_1} + |T_i| \mathbf{1}_{B_1 \leq r \leq 2B_1},$$

$$|b \partial_b \nabla \tilde{S}_i| = \left| b \chi_{B_1} \partial_b \nabla S_i - b \frac{\partial_b B_1}{B_1} (y \chi') \left(\frac{y}{B_1} \right) \nabla S_i \right| \lesssim |b \partial_b \nabla S_i| \mathbf{1}_{r \leq 2B_1} + |\nabla S_i| \mathbf{1}_{B_1 \leq r \leq 2B_1}.$$

Using with the bounds of the Proposition 3.1 yields (3.103), (3.104), (3.105), (3.106), (3.107), (3.108), (3.109) and (3.110).

Now, let's focus on the control of the error $\tilde{\Psi}_b$. In this purpose, we recall the definition of the partial mass of respectively Q_b and ΔP_b :

$$m_b(r) = \int_0^r Q_b(\tau) \tau d\tau, \quad n_b(r) = \int_0^r \Delta P_b(\tau) \tau d\tau.$$

Similarly, we define now the partial mass of respectively \tilde{Q}_b and $\Delta \tilde{P}_b$ by:

$$\tilde{m}_b(r) = m_0 + m_{\tilde{\alpha}_b} = \int_0^r \tilde{Q}_b(\tau) \tau d\tau,$$

$$\tilde{n}_b(r) = n_0 + n_{\tilde{\gamma}_b} = \int_0^r \Delta \tilde{P}_b(\tau) \tau d\tau.$$

Hence, we have :

$$m'_{\tilde{\alpha}_b} = \chi_{B_1} m'_{\alpha_b} \quad \text{and} \quad n_{\tilde{\gamma}_b} = \chi_{B_1} n_{\gamma_b}.$$

In the same way as follows, we can rewrite (3.116) by

$$\tilde{\Psi}_b = \left| \begin{array}{c} \tilde{\Psi}_b^{(1)} \\ \nabla \tilde{\Psi}_b^{(2)} \end{array} \right| = \frac{1}{r} \left| \begin{array}{c} \tilde{\Phi}'_b \\ \tilde{\Omega}'_b \end{array} \right| + c_b b^2 \check{\mathbf{T}} \quad (3.116)$$

where

$$\begin{aligned} \left| \begin{array}{c} \tilde{\Phi}_b \\ \tilde{\Omega}_b \end{array} \right| &= \left| \begin{array}{c} \tilde{m}_b'' - \frac{\tilde{m}_b'}{r} + \frac{\tilde{m}_b' \tilde{n}_b}{r^2} - br \tilde{m}_b' \\ (\tilde{n}_b - \tilde{m}_b)'' - \frac{(\tilde{n}_b - \tilde{m}_b)'}{r} - br \tilde{n}_b' \end{array} \right| \\ &= \left| \begin{array}{c} m_{\tilde{\alpha}_b}'' - \frac{m'_{\tilde{\alpha}_b}}{r} + \frac{m'_0 n_{\tilde{\gamma}_b} + m'_{\tilde{\alpha}_b} n_0 + m'_{\tilde{\alpha}_b} n_{\tilde{\gamma}_b}}{r^2} - br(m'_0 + m'_{\tilde{\alpha}_b}) \\ n_{\tilde{\gamma}_b}'' - m_{\tilde{\alpha}_b}'' - \frac{n'_{\tilde{\gamma}_b} - m'_{\tilde{\alpha}_b}}{r} - br(n'_0 + n'_{\tilde{\gamma}_b}) \end{array} \right| \\ &= \chi_{B_1} \left| \begin{array}{c} \Phi_b \\ \Omega_b \end{array} \right| + \left| \begin{array}{c} J_{1,b} \\ J_{2,b} \end{array} \right| \end{aligned} \quad (3.117)$$

where

$$\left| \begin{array}{c} J_{1,b} \\ J_{2,b} \end{array} \right| = -br(1 - \chi_{B_1}) \left| \begin{array}{c} m'_0 \\ n'_0 \end{array} \right| + \chi'_{B_1} \left| \begin{array}{c} m'_{\alpha_b} \\ -m'_{\alpha_b} + 2n'_{\gamma_b} - (br + \frac{1}{r}) n_{\gamma_b} \end{array} \right| + \left| \begin{array}{c} (\chi_{B_1}^2 - \chi_{B_1}) \frac{m'_{\alpha_b} n_{\gamma_b}}{r^2} \\ \chi_{B_1}'' n_{\gamma_b} \end{array} \right|$$

We recall that :

$$rm'_0 = rn'_0 = \frac{8r^2}{(1+r^2)^2}.$$

Now, with the bounds (3.44), and (3.82) :

$$|m'_{\alpha_b}| \lesssim \frac{b}{r} \mathbf{1}_{B_1 \leq r \leq 2B_1} + b^2 \frac{1}{r^3 b^2 |\log b|} \mathbf{1}_{B_1 \leq r \leq 2B_1} \lesssim \frac{b}{r} \mathbf{1}_{B_1 \leq r \leq 2B_1},$$

and with the estimate (3.46), (3.47) and (3.87), for $i \geq 0$:

$$|r^i \partial_r^i n_{\gamma_b}| \lesssim b \mathbf{1}_{B_1 \leq r \leq 2B_1} + b^2 r^2 \log r \mathbf{1}_{B_1 \leq r \leq 2B_1} \lesssim b^2 r^2 \log r \mathbf{1}_{B_1 \leq r \leq 2B_1}.$$

Using the following bounds, we obtain, for $i \geq 0$:

$$|r^i \partial_r^i J_{1,b}| \lesssim \frac{b}{r^2} \mathbf{1}_{r \geq B_1} \quad (3.118)$$

$$|r^i \partial_r^i J_{2,b}| \lesssim \frac{b}{r^2} \{ |\log b|^5 \mathbf{1}_{B_1 \leq r \leq 2B_1} + \mathbf{1}_{r \geq 2B_1} \} \quad (3.119)$$

Finally, from the definition (3.116) and (3.117) :

$$\begin{aligned} \tilde{\Psi}_b &= \frac{1}{r} \left| \begin{array}{c} \chi_{B_1} \Phi'_b + \chi'_{B_1} \Phi_b + J'_{1,b} \\ \chi_{B_1} \Omega_b + J_{2,b} \end{array} \right. + c_b b^2 \chi_{\frac{B_0}{4}} \left. \begin{array}{c} T_1 \\ \nabla S_1 \end{array} \right| \\ &= \chi_{B_1} \Psi_b + \frac{1}{r} \left| \begin{array}{c} \chi'_{B_1} \Phi_b + J'_{1,b} \\ J_{2,b} \end{array} \right| \end{aligned}$$

We now estimate from (3.94), (3.95), (3.118) and (3.119) :

$$\begin{aligned} |\tilde{\Psi}_b^{(1)} - \chi_{B_1} \Psi_b^{(1)}| &\lesssim \frac{b^2}{r^2} \mathbf{1}_{B_1 \leq r \leq 2B_1} + \frac{b}{r^4} \mathbf{1}_{r \geq B_1} \\ |\nabla \tilde{\Psi}_b^{(2)} - \chi_{B_1} \nabla \Psi_b^{(2)}| &\lesssim \frac{b}{r^3} \{ |\log b|^5 \mathbf{1}_{B_1 \leq r \leq 2B_1} + \mathbf{1}_{r \geq 2B_1} \} \end{aligned}$$

and hence using (3.33):

$$\int |r^i \partial_r^i \tilde{\Psi}_b^{(1)}|^2 \lesssim \frac{b^5}{|\log b|^2} + \int_{r \geq B_1} \left[\frac{b^4}{r^4} + \frac{b^2}{r^8} \right] \lesssim \frac{b^5}{|\log b|^2}.$$

Moreover :

$$\begin{aligned} \int \frac{|\mathcal{L}^{(1)}(\tilde{\Psi}_b^{(1)}, \tilde{\Psi}_b^{(2)})|^2}{Q} &\lesssim \sum_{i=0}^2 \int (1+r^{2i}) |\partial_r^i \tilde{\Psi}_b^{(1)}|^2 + \sum_{i=0}^1 \int \frac{|\partial_r^i \nabla \tilde{\Psi}_b^{(2)}|^2}{1+r^{6-2i}} \\ &\lesssim \frac{b^5}{|\log b|^2} + \int_{r \geq B_1} \left[\frac{b^4}{r^4} + \frac{b^2}{r^8} \right] + \int_{r \geq B_1} \frac{b^2 |\log b|^{10}}{r^{12}} \lesssim \frac{b^5}{|\log b|^2}. \end{aligned}$$

We estimate from (3.96)

$$\begin{aligned} \int \frac{|\nabla \tilde{\Psi}_b^{(2)}|^2}{1+r^2} + \int |\Delta \tilde{\Psi}_b^{(2)}|^2 &\lesssim b^5 |\log b|^2 + \int_{r \geq B_1} \frac{b^2 |\log b|^{10}}{r^8} \\ &\lesssim b^5 |\log b|^4 \end{aligned}$$

and from (3.97) :

$$\begin{aligned} \int |\nabla \tilde{\Psi}_b^{(2)}|^2 &\lesssim b^4 |\log b|^6 + \int_{r \geq B_1} \frac{b^2 |\log b|^{10}}{r^6} \\ &\lesssim b^4 |\log b|^6. \end{aligned}$$

To conclude this step of the proof :

$$\begin{aligned} \int Q |\nabla \mathcal{M}^{(1)}(\tilde{\Psi}_b^{(1)}, \tilde{\Psi}_b^{(2)})|^2 &\lesssim \int (1+r^2) \left[|r \partial_r \tilde{\Psi}_b^{(1)}|^2 + |\tilde{\Psi}_b^{(1)}|^2 \right] + \frac{|\nabla \tilde{\Psi}_b^{(2)}|^2}{1+r^4} \\ &\lesssim \frac{b^4}{|\log b|^2} + \int_{r \geq B_1} \left[\frac{b^4}{r^2} + \frac{b^2}{r^6} \right] + b^5 |\log b|^4 \\ &\lesssim \frac{b^4}{|\log b|^2}. \end{aligned}$$

Step 2 : Degenerate flux.

To prove the estimation of the degenerate flux, we use the rough bound (2.42) together the cancellation (2.27) to obtain :

$$\begin{aligned}
\left| \langle \mathcal{L}\tilde{\Psi}_b, \Phi_{0,B} \rangle \right| &\lesssim B \|\mathcal{L}\tilde{\Psi}_b\|_{X_Q}^2 \\
&\lesssim \frac{1}{\sqrt{b}} \left\{ \int \frac{|\mathcal{L}^{(1)}(\tilde{\Psi}_b^{(1)}, \tilde{\Psi}_b^{(2)})|^2}{Q} + \int \frac{|\Delta \tilde{\Psi}_b^{(2)}|^2}{1+r^2} + \int |\tilde{\Psi}_b^{(1)}|^2 \right\} \\
&\lesssim \frac{b^{\frac{5}{2}}}{\sqrt{b}|\log b|} \lesssim \frac{b^2}{|\log b|}.
\end{aligned}$$

This concludes the proof of the Proposition 3.2. \square

4. Control of the perturbation of the approximate profile

After the construction of approximate profile in the above section, the goal of the rest of this paper is to prove the existence of an open set \mathcal{O} of initial data, such that we can split the corresponding solution to the problem (1.2) in two parts. The first is the approximate profile. The second is an error term, which doesn't perturb the blow-up dynamics whose we have formally predicted by the construction. In this purpose, we use a bootstrap argument implementing an energy method.

4.1. On the open set of initial data. In this subsection, we describe the open set \mathcal{O} of initial data, whose corresponding strong solution to (1.2) satisfy the Theorem 1.1.

To begin, the following lemma gives the uniqueness of the decomposition of the solution, under orthogonality conditions.

Lemma 4.1. *$L^1 \times \dot{H}^1$ modulation Let $M > M^*$ large enough, then there exists a universal constant $\delta^*(M) > 0$ such that $\forall (v, w) \in L^1 \times \dot{H}^1$ with*

$$\|v - Q\|_{L^1} + \|\nabla w - \nabla \phi_Q\|_{L^2} < \delta^*(M),$$

there exists a unique decomposition

$$\begin{vmatrix} v \\ \nabla w \end{vmatrix} = \begin{vmatrix} \frac{1}{\lambda_1^2} (\tilde{Q}_b + \varepsilon_1) \left(\frac{r}{\lambda_1} \right) \\ \frac{1}{\lambda_1} (\nabla \tilde{P}_b + \nabla \eta_1) \left(\frac{r}{\lambda_1} \right) \end{vmatrix}$$

such that

$$\langle \varepsilon_1, \Phi_M \rangle = \langle \varepsilon_1, \mathcal{L}^* \Phi_M \rangle = 0.$$

Moreover,

$$\|\varepsilon_1\|_{L^1} + \|\nabla \eta_1\|_{L^2} + |\lambda_1 - 1| + |b| \lesssim c(M)\delta^*.$$

Proof. To prove this lemma, we use the implicit function theorem. Indeed, consider the \mathcal{C}^1 functional

$$F(v, \lambda_1, b) = [\langle \mathbf{v}_{Mod}, \Phi_M \rangle, \langle \mathbf{v}_{Mod}, \mathcal{L}^* \Phi_M \rangle],$$

where

$$\mathbf{v}_{Mod} = \begin{vmatrix} v_{Mod} \\ \nabla w_{Mod} \end{vmatrix} = \begin{vmatrix} \lambda_1^2 v(\lambda_1 x) - \tilde{Q}_b \\ \lambda_1 \nabla w(\lambda_1 x) - \nabla \tilde{P}_b \end{vmatrix}$$

and

$$Mod = (\lambda_1, b).$$

By definition, we have $F(\mathbf{Q}, 1, 0) = 0$. Now, we compute the Jacobian at this point, using (2.37):

$$\begin{aligned} & \left| \begin{array}{cc} \left\langle \frac{\partial}{\partial \lambda_1}(\mathbf{v}_{Mod}), \Phi_M \right\rangle & \left\langle \frac{\partial}{\partial b}(\mathbf{v}_{Mod}), \Phi_M \right\rangle \\ \left\langle \frac{\partial}{\partial \lambda_1}(\mathbf{v}_{Mod}), \mathcal{L}^* \Phi_M \right\rangle & \left\langle \frac{\partial}{\partial b}(\mathbf{v}_{Mod}), \mathcal{L}^* \Phi_M \right\rangle \end{array} \right|_{(\mathbf{v}, Mod)=(Q, 1, 0)} \\ &= \left| \begin{array}{cc} \langle -\Lambda \mathbf{Q}, \Phi_M \rangle & 0 \\ 0 & \langle \mathbf{T}_1, H \Phi_M \rangle \end{array} \right| \end{aligned} \quad (4.1)$$

$$= -\langle \Lambda \mathbf{Q}, \Phi_M \rangle^2 = (-32\pi \log M + O(1))^2 > 0, \quad (4.2)$$

for M large enough. We can apply the implicit function theorem and this concludes the proof of the lemma 4.1. \square

Now, we are in position to describe a open set \mathcal{O} of initial data u_0 whose the corresponding strong solution to (1.2) satisfy the dynamics describes in the Theorem 1.1.

Definition 4.2 (Description of the open set \mathcal{O} of initial data). *Let $M > M^*$ large enough. Let $\alpha^*(M)$ small enough. Pick α^* such that $0 < \alpha^* < \alpha^*(M)$. Then, we let \mathcal{O} the set of initial data of the form*

$$\begin{cases} u_0 \\ v_0 \end{cases} = \begin{cases} \frac{1}{\lambda_0^2} \{ \tilde{Q}_{b_0} + \varepsilon_0 \} \left(\frac{r}{\lambda_0} \right) \\ \{ \tilde{P}_{b_0} + \eta_0 \} \left(\frac{r}{\lambda_0} \right) \end{cases} \quad (4.3)$$

where the perturbation (ε_0, η_0) satisfies :

- Orthogonality conditions

$$\langle \varepsilon_0, \Phi_M \rangle = \langle \varepsilon_0, \mathcal{L}^* \Phi_M \rangle = 0. \quad (4.4)$$

- Positivity:

$$u_0 > 0, \quad v_0 > 0.$$

- Small super critical mass:

$$\int Q < \int u_0 < \int Q + \alpha^*. \quad (4.5)$$

- Positivity and smallness of b_0 :

$$0 < b_0 < \delta(\alpha^*). \quad (4.6)$$

- Initial smallness:

$$\|(1+y)\nabla \Delta \eta_0\|_{L^2} + \|\Delta \eta_0\|_{L^2} + \|\eta_0\|_{\dot{H}^1} < b_0^{10} \quad (4.7)$$

$$\|\varepsilon_0\|_{X_Q} < b_0^{10}. \quad (4.8)$$

Remark 4.3. We choose $\alpha^*(M)$ small enough in order to have the uniqueness of the decomposition (4.3).

4.2. The bootstrap argument. Let $\mathbf{u}_0 \in \mathcal{O}$, and $\mathbf{u} \in \mathcal{C}([0, T[, \mathcal{E})$ be the corresponding strong solution to (1.2). As \mathbf{u}_0 is a very small perturbation in $L^1 \times \dot{H}^1$ of the soliton, we can apply the lemma (4.1), and thus, the solution \mathbf{u} admits a unique decomposition on some small time interval $[0, T^*)$

$$\begin{cases} u(t, r) \\ v(t, r) \end{cases} = \begin{cases} \frac{1}{\lambda(t)^2} \{ \tilde{Q}_{b(t)} + \varepsilon(t) \} \left(\frac{r}{\lambda(t)} \right) \\ \{ \tilde{P}_{b(t)} + \eta(t) \} \left(\frac{r}{\lambda(t)} \right) \end{cases} \quad (4.9)$$

where the error term $\varepsilon(t)$ satisfies the orthogonality conditions

$$\langle \varepsilon(t), \Phi_M \rangle = \langle \varepsilon(t), \mathcal{L}^* \Phi_M \rangle = 0. \quad (4.10)$$

Moreover, using a similar argument that Martel and Merle in [27], we prove that the geometrical parameters $(\lambda(t), b(t))$ are nonnegative continuous function.

The goal of the rest of the paper is to prove that the error term $\varepsilon(t)$ doesn't perturb the blow-up dynamics formally predicted by the construction of the approximate profile $\tilde{\mathbf{Q}}_b$. In this purpose, we shall use a bootstrap argument to prove that the error term remains very small in suitable norms with respect to $b(t)$. Thus, we shall obtain bounds on the error made on the equation of modulation parameters (λ, b) . We shall be in position to conclude the proof of the Theorem 1.1. Using the continuity of geometrical parameters together with the initial smallness assumption, we may assume on $[0, T^*)$ the bootstrap bounds :

- Positivity and smallness of b :

$$0 < b(t) < 10b_0. \quad (4.11)$$

- L^1 bound:

$$\int |\varepsilon(t)| < (\delta^*)^{\frac{1}{4}}. \quad (4.12)$$

- Control of ε in smoother norms:

$$\|\eta(t)\|_{\dot{H}^1}^2 \leq K^* b^2(t) |\log b(t)|^6, \quad (4.13)$$

$$\|\Delta \eta(t)\|_{L^2}^2 \leq K^* \frac{b^2(t)}{|\log b(t)|}, \quad (4.14)$$

$$\|(1+y)\nabla \Delta \eta(t)\|_{L^2}^2 \leq K^* \frac{b(t)}{\sqrt{|\log b(t)|}}, \quad (4.15)$$

$$\|\mathbf{E}_2(t)\|_{X_Q}^2 \leq K^* \frac{b^3(t)}{|\log b(t)|^2}, \quad (4.16)$$

where $\mathbf{E}_2 = \mathcal{L}(\varepsilon, \eta)$, $K^* = K^*(M)$ is a large enough constant and $\delta^* = \delta(\alpha^*)$ is a small enough constant such that:

$$\delta(\alpha^*) \rightarrow 0 \text{ as } \alpha^* \rightarrow 0.$$

The following proposition prove that the regime is trapped, and thus $T^* = T$.

Proposition 4.4 (Trapped regime). *Assume that K^* in (4.16) has been chosen large enough, then $\forall t \in [0, T^*)$:*

$$0 < b(t) < 2b_0, \quad (4.17)$$

$$\int |\varepsilon(t)| < \frac{1}{2}(\delta^*)^{\frac{1}{4}}, \quad (4.18)$$

$$\|\eta(t)\|_{\dot{H}^1}^2 \leq \frac{K^*}{2} b^2(t) |\log b(t)|^6, \quad (4.19)$$

$$\|\Delta \eta(t)\|_{L^2}^2 \leq \frac{K^*}{2} \frac{b^2(t)}{|\log b(t)|}, \quad (4.20)$$

$$\|(1+y)\nabla \Delta \eta(t)\|_{L^2}^2 \leq \frac{K^*}{2} \frac{b(t)}{\sqrt{|\log b(t)|}}, \quad (4.21)$$

$$\|\mathbf{E}_2(t)\|_{X_Q}^2 \leq \frac{K^*}{2} \frac{b^3(t)}{|\log b(t)|^2}. \quad (4.22)$$

The rest of the paper is organized as follows : In the rest of this section, we shall set up the equation verified by the error term, and we compute the rough modulation equation, coming from our choice of orthogonality conditions (4.10). We shall introduce a second decomposition for technical reasons, and we shall be in position to obtain a sharp modulation equation. The section 5 is devoted to the proof of the suitable Lyapounov functional at the X_Q level, which is the heart of the energy method to prove (4.22). Finally, we shall prove the Proposition 4.4 in the section 6.1. We shall have therefore all tools to prove the Theorem 1.1 in the last section 7.

4.3. Equation of the error term. We recall the space time renormalization :

$$s(t) = \int_0^t \frac{d\tau}{\lambda^2(\tau)}, \quad y = \frac{r}{\lambda(t)}. \quad (4.23)$$

and the notation

$$\mathbf{f}_\lambda(t, r) = \begin{vmatrix} f_\lambda(t, r) \\ g_\lambda(t, r) \end{vmatrix} = \begin{vmatrix} \frac{1}{\lambda^2} f(s, y) \\ g(s, y) \end{vmatrix} \quad (4.24)$$

which leads to:

$$\partial_t \begin{vmatrix} f_\lambda \\ g_\lambda \end{vmatrix} = \frac{1}{\lambda^2} \begin{vmatrix} [\partial_s f - \frac{\lambda_s}{\lambda} \Lambda f]_\lambda \\ [\partial_s g - \frac{\lambda_s}{\lambda} \Lambda g]_\lambda \end{vmatrix}. \quad (4.25)$$

Let the renormalized flow :

$$\begin{vmatrix} u_\lambda(t, r) \\ v_\lambda(t, r) \end{vmatrix} = \begin{vmatrix} f_\lambda(s, y) \\ g_\lambda(s, y) \end{vmatrix}.$$

Then if (u, v) is a solution of the problem (1.2), (f, g) satisfy the following system :

$$\begin{cases} \partial_s f - \frac{\lambda_s}{\lambda} \Lambda f = \nabla \cdot (\nabla f + f \nabla g) \\ \partial_s g - \frac{\lambda_s}{\lambda} \Lambda g = \Delta g - f \end{cases}$$

Using the unique decomposition (4.9) for the solutions whose initial data are in \mathcal{O} , we decompose (f, g) as following

$$\begin{vmatrix} f(s, y) \\ g(s, y) \end{vmatrix} = \begin{vmatrix} \tilde{\mathbf{Q}}_{b(s)} + \mathbf{E}(s, y) \\ \tilde{P}_{b(s)} + \eta(s, y) \end{vmatrix}. \quad (4.26)$$

Hence \mathbf{E} is solution of the following equation:

$$\partial_s \mathbf{E} - \frac{\lambda_s}{\lambda} \Lambda \mathbf{E} = \mathcal{L} \mathbf{E} + \mathbf{F} + \widetilde{\mathbf{Mod}} + \mathbf{G} = \mathcal{L} \mathbf{E} + \mathcal{F}, \quad (4.27)$$

where

$$\mathbf{F} = \tilde{\Psi}_b + \Theta_b(\varepsilon, \eta) + \mathbf{N}(\varepsilon, \eta) \quad (4.28)$$

with $\tilde{\Psi}_b$ is defined by (3.116) and

$$\Theta_b(\varepsilon, \eta) = \begin{vmatrix} \nabla \cdot (\varepsilon \nabla \tilde{\gamma}_b + \tilde{\alpha}_b \nabla \eta) \\ 0 \end{vmatrix}, \quad (4.29)$$

$$\mathbf{N}(\varepsilon, \eta) = \begin{vmatrix} \nabla \cdot (\varepsilon \nabla \eta) \\ 0 \end{vmatrix}, \quad (4.30)$$

$$\widetilde{\mathbf{Mod}}(y, s) = \left(b + \frac{\lambda_s}{\lambda} \right) \Lambda \tilde{\mathbf{Q}}_b - \partial_s \tilde{\mathbf{Q}}_b, \quad (4.31)$$

$$\mathbf{G} = -c_b b^2 \check{\mathbf{T}}. \quad (4.32)$$

In the same way, using the original variables, we can rewrite the decomposition by

$$\mathbf{U} = \left(\tilde{\mathbf{Q}}_b \right)_\lambda + \mathbf{W} = \left(\begin{vmatrix} Q + \tilde{\alpha}_b \\ \phi_Q + \tilde{\gamma}_b \end{vmatrix} \right)_\lambda + \begin{vmatrix} w \\ z \end{vmatrix}. \quad (4.33)$$

Hence \mathbf{W} verifies :

$$\partial_t \mathbf{W} = \mathcal{L}_\lambda \mathbf{W} + \frac{1}{\lambda^2} \mathcal{F}_\lambda \quad (4.34)$$

with

$$\mathcal{L}_\lambda \mathbf{W} = \begin{vmatrix} \nabla \cdot (Q_\lambda \nabla \mathcal{M}_\lambda^{(1)}(w, z)) \\ \Delta \mathcal{M}_\lambda^{(2)}(w, z) \end{vmatrix}, \quad \mathcal{M}_\lambda(\mathbf{W}) = \begin{vmatrix} \frac{w}{Q_\lambda} + z \\ w - \phi_z \end{vmatrix} \quad (4.35)$$

4.4. First modulation equations. We are in position to compute the two modulation equation of (b, λ) , projecting the equation (4.27) respectively onto Φ_M and $\mathcal{L}\Phi_M$.

Lemma 4.5 (Rough control of the modulation parameters). *There exist an universal constant $C(M)$ large enough, independent of the bootstrap constant $K^*(M)$ such that:*

$$\left| \frac{\lambda_s}{\lambda} + b \right| \lesssim C(M) \frac{b^2}{|\log b|}, \quad (4.36)$$

$$|b_s| \lesssim b^{\frac{3}{2}}. \quad (4.37)$$

Remark 4.6. Note that (4.36), (4.37) imply the bootstrap bound:

$$\left| \frac{\lambda_s}{\lambda} + b \right| + |b_s| \lesssim b^{\frac{3}{2}}. \quad (4.38)$$

Proof. Let

$$V = \left| \frac{\lambda_s}{\lambda} + b \right| + |b_s| \quad (4.39)$$

Step 1 : Projection onto Φ_M

We project the equation (4.27) onto Φ_M . Using the orthogonality conditions (4.10), we obtain:

$$-\frac{\lambda_s}{\lambda} \langle \Lambda \mathbf{E}, \Phi_M \rangle = \langle \mathbf{F}, \Phi_M \rangle + \langle \widetilde{\mathbf{Mod}}, \Phi_M \rangle + \langle \mathbf{G}, \Phi_M \rangle. \quad (4.40)$$

From (2.35), the function $\Phi_M^{(1)}$ and $\nabla \phi_M^{(2)}$ are compactly supported in $r \leq 2M$. Using the interpolation bound (C.1) and the bootstrap bound (4.16), we obtain:

$$\begin{aligned} \left| -\frac{\lambda_s}{\lambda} \langle \Lambda \mathbf{E}, \Phi_M \rangle \right| &\lesssim (b+V) \left\{ \int \nabla \cdot (y\varepsilon) \Phi_M^{(1)} + \int \nabla (y \nabla \eta) \nabla \Phi_M^{(2)} \right\} \\ &\lesssim (b+V) C(M) \left\{ \left(\int \varepsilon^2 \right)^{\frac{1}{2}} + \left(\int \frac{|\nabla \eta|^2}{1+y^4} \right)^{\frac{1}{2}} \right\} \\ &\lesssim (b+V) C(M) \frac{b^{\frac{3}{2}}}{|\log b|}. \end{aligned}$$

Now from the bound (3.111) and (3.112),

$$\begin{aligned} |\langle \Psi_b, \Phi_M \rangle| &= \left| \int \tilde{\Psi}_b^{(1)} \Phi_M^{(1)} + \int \nabla \tilde{\Psi}_b^{(2)} \nabla \Phi_M^{(2)} \right| \\ &\lesssim C(M) \left[\left(\int |\tilde{\Psi}_b^{(1)}|^2 \right)^{\frac{1}{2}} + \left(\int \frac{|\nabla \tilde{\Psi}_b^{(2)}|^2}{1+y^2} \right)^{\frac{1}{2}} \right] \\ &\lesssim C(M) b^{\frac{5}{2}} |\log b|^2 \end{aligned}$$

The bootstrap bound (4.16) yields :

$$|\langle \mathbf{N}, \Phi_M \rangle| = \left| \int \nabla \cdot (\varepsilon \nabla \eta) \Phi_M^{(1)} \right| \lesssim C(M) \left(\int \varepsilon^2 \int \frac{|\nabla \eta|^2}{1+y^4} \right)^{\frac{1}{2}} \lesssim C(M) \frac{b^{\frac{3}{2}}}{|\log b|}.$$

In the same way, we prove that : $|\langle \Theta_b, \Phi_M \rangle| \lesssim C(M) \frac{b^{\frac{3}{2}}}{|\log b|}$. Now, using the definition of Φ_M (2.35), and the decay of T_1 (3.103), we obtain :

$$|\langle \mathbf{G}, \Phi_M \rangle| = \left| c_b b^2 \left\{ \int T_1 \chi_{\frac{B_0}{4}} \Phi_M^{(1)} + \int \nabla S_1 \chi_{\frac{B_0}{4}} \nabla \Phi_M^{(2)} \right\} \right| \lesssim C(M) \frac{b^2}{|\log b|}.$$

Finally, using the estimate on Φ_M (2.37), we obtain :

$$\begin{aligned} \left(b + \frac{\lambda_s}{\lambda} \right) \langle \Lambda \tilde{\mathbf{Q}}_b, \Phi_M \rangle &= \left(b + \frac{\lambda_s}{\lambda} \right) \langle \Lambda \mathbf{Q}, \Phi_M \rangle + O(C(M)bV) \\ &= \left(b + \frac{\lambda_s}{\lambda} \right) (-32\pi \log M + O(1)) + O(C(M)bV), \end{aligned}$$

and

$$\left| \langle \partial_s \tilde{\mathbf{Q}}_b, \Phi_M \rangle \right| = \left| b_s \langle \tilde{\mathbf{T}}_1, \Phi_M \rangle + 2bb_s \langle \tilde{\mathbf{T}}_2, \Phi_M \rangle \right| \lesssim C(M)bV.$$

Hence, injecting all above bounds in (4.40), we obtain the bound

$$\left| \frac{\lambda_s}{\lambda} + b \right| \lesssim b^{\frac{3}{4}} |V| + C(M) \frac{b^2}{|\log b|}. \quad (4.41)$$

Step 2 : Projection onto $\mathcal{L}^* \Phi_M$

We project the equation (4.27) onto $\mathcal{L}^* \Phi_M$. Using the orthogonality conditions (4.10), we obtain:

$$-\frac{\lambda_s}{\lambda} \langle \Lambda \mathbf{E}, \mathcal{L}^* \Phi_M \rangle = \langle \mathbf{E}_2, \mathcal{L}^* \Phi_M \rangle + \langle \mathbf{F}, \mathcal{L}^* \Phi_M \rangle + \langle \widetilde{\mathbf{Mod}}, \mathcal{L}^* \Phi_M \rangle + \langle \mathbf{G}, \mathcal{L}^* \Phi_M \rangle. \quad (4.42)$$

In the same way as the last step, we prove without difficulties that :

$$\left| \left\langle -\frac{\lambda_s}{\lambda} \Lambda \mathbf{E} - \mathbf{F} - \mathbf{G}, \mathcal{L}^* \Phi_M \right\rangle \right| \lesssim C(M) \left(bV + \frac{b^2}{|\log b|} \right).$$

The bound (2.43) and the bootstrap bound (4.16) imply:

$$|\langle \mathbf{E}_2, \mathcal{L}^* \Phi_M \rangle| \lesssim \frac{\|\mathbf{E}_2\|_{X_Q}}{M} \lesssim b^{\frac{3}{2}}.$$

We recall that $\mathcal{L} \Lambda \mathbf{Q} = 0$. Hence :

$$\left| \left(b + \frac{\lambda_s}{\lambda} \right) \langle \Lambda \tilde{\mathbf{Q}}_b, \mathcal{L}^* \Phi_M \rangle \right| \lesssim Vb \left| \langle \mathcal{L} \Lambda \tilde{\mathbf{T}}_1 + b \mathcal{L} \Lambda \tilde{\mathbf{T}}_2, \Phi_M \rangle \right| \lesssim C(M)bV.$$

To conclude,

$$\begin{aligned} \langle \partial_s \tilde{\mathbf{Q}}_b, \mathcal{L}^* \Phi_M \rangle &= b_s \langle \mathcal{L} \tilde{\mathbf{T}}_1, \Phi_M \rangle + O(C(M)bV) \\ &= b_s (-32\pi \log M + O(1)) + O(C(M)bV). \end{aligned}$$

Injecting the collection of above estimates in (4.42) yields :

$$|b_s| \lesssim b^{\frac{3}{2}} + C(M)b|V|. \quad (4.43)$$

Step 3 : Conclusion

Summing the bound (4.41) and (4.43) yields for b enough small

$$V \lesssim b^{\frac{3}{2}}. \quad (4.44)$$

Injecting this bound in (4.41) and (4.43) concludes the proof of the bounds (4.36) and (4.37) and thus the proof of the Lemma 4.5. \square

4.5. Second decomposition of the flow. By the construction, we have formally predicted that the law of b verify $b_s = -2\frac{b^2}{|\log b|}(1 + O(1))$. The bound (4.37) isn't enough good to know precisely the gap between b_s and $-2\frac{b^2}{|\log b|}$. This is a consequence of the too slow decay of the elements of the kernel of \mathcal{L}^* . Thus, as in [36], we introduce a second decomposition of the flow, which will lift the parameter b .

Lemma 4.7 (Second decomposition). *There exists a unique decomposition*

$$\tilde{\mathbf{Q}}_b + \mathbf{E} = \tilde{\mathbf{Q}}_{\hat{b}} + \hat{\mathbf{E}} = \tilde{\mathbf{Q}}_{\hat{b}} + \begin{vmatrix} \hat{\varepsilon} \\ \hat{\eta} \end{vmatrix} \quad (4.45)$$

such that $\hat{b} \lesssim b \lesssim \hat{b}$ and $\hat{\varepsilon}$ satisfying the orthogonality condition

$$\langle \hat{\mathbf{E}}, \mathcal{L}^* \Phi_{0, \hat{B}_0} \rangle = 0, \quad \hat{B}_0 = \frac{1}{\sqrt{\hat{b}}}. \quad (4.46)$$

Moreover, there holds the bound:

$$|b - \hat{b}| \lesssim \frac{b}{|\log b|}. \quad (4.47)$$

Proof. Let the map $F(\mathbf{V}, \hat{b}) = \langle \mathbf{V} - \tilde{\mathbf{Q}}_{\hat{b}}, \mathcal{L}^* \Phi_{0, \hat{B}_0} \rangle$, for $|b - \hat{b}| \lesssim b$. First, remark that:

$$F(\tilde{\mathbf{Q}}_b, b) = 0. \quad (4.48)$$

Moreover,

$$\frac{\partial F}{\partial \hat{b}} \Big|_{(\hat{b}=b, \mathbf{V}=\tilde{\mathbf{Q}}_b)} = - \left\langle \frac{\partial \tilde{\mathbf{Q}}_{\hat{b}}}{\partial \hat{b}}, \mathcal{L}^* \Phi_{0, B_0} \right\rangle. \quad (4.49)$$

Using the bounds of the Proposition 3.2, we obtain:

$$\begin{aligned} \frac{\partial P_{B_1}}{\partial b} &= \chi_{B_1} T_1 + O \left(\frac{1}{1+r^2} \mathbf{1}_{B_1 \leq r \leq 2B_1} + b \left[\frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{r \leq 6B_0} + \frac{1}{b^2 r^4 |\log b|} \mathbf{1}_{6B_0 \leq r \leq 2B_1} \right] \right), \\ \frac{\partial \nabla \tilde{P}_b}{\partial b} &= \chi_{B_1} \nabla S_1 + O \left(\frac{1}{1+r} \mathbf{1}_{B_1 \leq r \leq 2B_1} + br(1 + \log r)^2 \mathbf{1}_{1 \leq r \leq 2B_1} \right). \end{aligned}$$

Hence, these bounds together (2.39) yield

$$\begin{aligned} & \left(\frac{\partial P_{B_1}}{\partial b}, (\mathcal{L}^*)^{(1)} \left(\Phi_{0, B_0}^{(1)}, \Phi_{0, B_0}^{(2)} \right) \right) \\ &= \left(\chi_{B_1} T_1, (\mathcal{L}^*)^{(1)} \left(\Phi_{0, B_0}^{(1)}, \Phi_{0, B_0}^{(2)} \right) \right) + bO \left(\int \left[\frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{r \leq 6B_0} + \frac{1}{b^2 r^4 |\log b|} \mathbf{1}_{6B_0 \leq r \leq 2B_1} \right] \right) \end{aligned}$$

and

$$\begin{aligned} & \left(\frac{\partial \nabla \tilde{P}_b}{\partial b}, \nabla (\mathcal{L}^*)^{(2)} \left(\Phi_{0, B_0}^{(1)}, \Phi_{0, B_0}^{(2)} \right) \right) \\ &= O \left(\int \frac{|\log b|}{1+r^4} \mathbf{1}_{r \leq 2B_1} \right) + b|\log b|O \left(\int_{r \leq 2B_1} \frac{r(1 + \log r)^2}{1+r^2} \right) = O(1). \end{aligned}$$

Thus with (2.37) :

$$\begin{aligned} \left\langle \frac{\partial \tilde{\mathbf{Q}}_{\hat{b}}}{\partial b}, \mathcal{L}^* \Phi_{0,B_0} \right\rangle &= \left(\frac{\partial P_{B_1}}{\partial b}, (\mathcal{L}^*)^{(1)} \left(\Phi_{0,B_0}^{(1)}, \Phi_{0,B_0}^{(2)} \right) \right) + \left(\frac{\partial \nabla \tilde{P}_b}{\partial b}, \nabla (\mathcal{L}^*)^{(2)} \left(\Phi_{0,B_0}^{(1)}, \Phi_{0,B_0}^{(2)} \right) \right) \\ &= -32\pi \log B_0 + O(1) < 0. \end{aligned}$$

From the implicit function theorem, this concludes the proof of the existence and the uniqueness of the decomposition (4.45) with $b \sim \hat{b}$.

Now, we let prove (4.47). We claim the following bound:

$$\left\langle \tilde{\mathbf{Q}}_b - \mathbf{Q}, \mathcal{L}^* \Phi_{0,B} \right\rangle = -32\pi b \log B + O(b), \quad \text{for } \frac{1}{\sqrt{b}} \lesssim B \lesssim \frac{1}{\sqrt{b}}. \quad (4.50)$$

Let take the scalar product of (4.45) with $\mathcal{L}^* \Phi_{0,\hat{B}_0}$.

$$\left\langle \tilde{\mathbf{Q}}_b - \tilde{\mathbf{Q}}_{\hat{b}}, \mathcal{L}^* \Phi_{0,\hat{B}_0} \right\rangle = \left\langle \hat{\mathbf{E}} - \mathbf{E}, \mathcal{L}^* \Phi_{0,\hat{B}_0} \right\rangle$$

With the bound (4.50), we have:

$$\left\langle \tilde{\mathbf{Q}}_b - \tilde{\mathbf{Q}}_{\hat{b}}, \mathcal{L}^* \Phi_{0,\hat{B}_0} \right\rangle = 32\pi \log \hat{B}_0 (b - \hat{b}) + O(b + \hat{b}).$$

The orthogonality condition (4.46) yields:

$$|\log b| |b - \hat{b}| \lesssim |b| + |\hat{b}| + \left| \left\langle \mathbf{E}_2, \Phi_{0,\hat{B}_0} \right\rangle \right| \lesssim |b| + \frac{\|\mathbf{E}_2\|_{L^2_Q}}{\sqrt{b}}$$

The last inequality together with the bootstrap bound (4.16), which we recall here:

$$\|\mathbf{E}_2\|_{X_Q} \lesssim \frac{b^3}{|\log b|^2}.$$

conclude the proof of the lemma 4.7.

Proof of (4.50):

To begin :

$$\begin{aligned} \left\langle \tilde{\mathbf{Q}}_b - \mathbf{Q}, \mathcal{L}^* \Phi_{0,B} \right\rangle &= \left\langle \tilde{\Upsilon}_b, \mathcal{L}^* \Phi_{0,B} \right\rangle \\ &= b \left\langle \mathcal{L} \mathbf{T}_1, \Phi_{0,B} \right\rangle + \left\langle b(\tilde{\mathbf{T}}_1 - \mathbf{T}_1) + b^2 \tilde{\mathbf{T}}_2, \mathcal{L}^* \Phi_{0,B} \right\rangle \end{aligned}$$

By contruction, and using (2.37), we have:

$$b \left\langle \mathcal{L} \mathbf{T}_1, \Phi_{0,B} \right\rangle = b \left\langle \Lambda \mathbf{Q}, \Phi_{0,B} \right\rangle = -32\pi b \log B + O(b).$$

Now, Now, from (2.39):

$$\begin{aligned} \left| \left\langle b^2 \tilde{\mathbf{T}}_2, \mathcal{L}^* \Phi_{0,B} \right\rangle \right| &\lesssim b^2 \int_{r \leq 2B_1} \left[r^4 \mathbf{1}_{r \leq 1} + \frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^2 r^4 |\log b|} \mathbf{1}_{6B_0 \leq r \leq 2B_1} \right] \\ &+ b^2 |\log b| \int_{1 \leq r \leq 2B_1} \frac{r(1 + \log r)}{1 + r^3} \\ &\lesssim b \end{aligned}$$

This concludes the proof of (4.50). \square

Let

$$\Xi = \begin{vmatrix} \zeta \\ \xi \end{vmatrix} = \hat{\mathbf{E}} - \mathbf{E} = \tilde{\mathbf{Q}}_b - \tilde{\mathbf{Q}}_{\hat{b}} = \begin{vmatrix} \tilde{\alpha}_b - \tilde{\alpha}_{\hat{b}} \\ \tilde{\gamma}_b - \tilde{\gamma}_{\hat{b}} \end{vmatrix} \quad (4.51)$$

the gap between the two decompositions which we split in two parts :

$$\Xi = \Xi_{big} + \Xi_{sm}, \quad (4.52)$$

with

$$\Xi_{big} = \begin{vmatrix} \zeta_{big} \\ \xi_{big} \end{vmatrix} = (b - \hat{b})\tilde{\mathbf{T}}_1, \quad (4.53)$$

$$\Xi_{sm} = \begin{vmatrix} \zeta_{sm} \\ \xi_{sm} \end{vmatrix} = \hat{b} \left| \int_0^r (\chi_{B_1} - \chi_{\hat{B}_1}) T_1 \right. + \int_{\hat{b}}^b \left[2b\tilde{\mathbf{T}}_2 + b^2\partial_b\tilde{\mathbf{T}}_2 \right] db \quad (4.54)$$

It's important to remark that Ξ_{big} is supported along T_1 . Hence, we will use this particular structure to improve some degenerate norms, in order to close the bootstrap. Let:

$$\Xi_2 = \mathcal{L}\Xi. \quad (4.55)$$

The following lemma describes adapted bounds on the gap between the two decomposition :

Lemma 4.8 (Control of the gap). *There holds the pointwise bounds:*

- *Estimates on Ξ_{big} :*

$$\begin{aligned} & \int |\tau^i \partial_\tau^i \zeta_{big}|^2 + \int \frac{|\nabla \phi_{\zeta_{big}}|^2}{1 + \tau^2} + \int \frac{|\tau^i \partial_\tau^i \nabla \xi_{big}|^2}{1 + \tau^2} \\ & \quad + \|\mathcal{L}(\Xi_{big})\|_{X_Q}^2 + \|\mathcal{L}(\Lambda \Xi_{big})\|_{X_Q}^2 \lesssim \frac{b^2}{|\log b|^2}. \end{aligned} \quad (4.56)$$

- *Estimates on Ξ_{sm} :*

$$\begin{aligned} & \int |\tau^i \nabla^i \zeta_{sm}|^2 + \int \frac{|\nabla \phi_{\zeta_{sm}}|^2}{1 + \tau^2} + \int \frac{|\tau^i \partial_\tau^i \xi_{sm}|^2}{1 + \tau^2} \\ & \quad + \|\mathcal{L}(\Xi_{sm})\|_{X_Q}^2 + \|\mathcal{L}(\Lambda \Xi_{sm})\|_{X_Q}^2 \lesssim \frac{b^3}{|\log b|^2}, \end{aligned} \quad (4.57)$$

$$\int \left| \mathcal{L}^{(1)}(\zeta_{sm}, \xi_{sm}) \right|^2 + \int \frac{|\mathcal{L}^{(2)}(\zeta_{sm}, \xi_{sm})|^2}{1 + \tau^2} \lesssim \frac{b^4}{|\log b|^2}. \quad (4.58)$$

- *Estimates on Ξ :*

$$\int |\nabla \xi|^2 \lesssim b^2 |\log b|^2 \quad (4.59)$$

$$\int Q |\nabla \mathcal{M}^{(1)} \Lambda \Xi|^2 + \int |\Delta \mathcal{M}^{(2)} \Lambda \Xi|^2 \lesssim \frac{b^2}{|\log b|^2}. \quad (4.60)$$

- *Estimates on Ξ_2 : Let $\mathbf{v} = (v_1, v_2) \in L_Q^2 \times \dot{H}^1$ such that $\int v_1 = 0$, then:*

$$\left| \int \mathcal{M}^{(1)}(\zeta_2, \xi_2) v_1 \right| \lesssim \frac{b^{\frac{3}{2}}}{|\log b|} \|v_1\|_{L_Q^2}, \quad (4.61)$$

$$|\langle \mathcal{M} \Xi_2, \mathbf{v} \rangle| \lesssim \frac{b^{\frac{3}{2}}}{|\log b|} \|\mathbf{v}\|_{X_Q}, \quad (4.62)$$

Moreover,

$$\left| \int \mathcal{M}^{(1)}(\zeta_2, \xi_2) \zeta_2 \right| \lesssim \frac{b^3}{|\log b|^2}, \quad (4.63)$$

$$|\langle \mathcal{M} \Xi_2, \Xi_2 \rangle| \lesssim \frac{b^3}{|\log b|^2}, \quad (4.64)$$

$$\int Q |\nabla \mathcal{M}^{(1)} \Xi_2|^2 + \int |\Delta \mathcal{M}^{(2)} \Xi_2|^2 \lesssim \frac{b^4}{|\log b|^2}. \quad (4.65)$$

Proof. First, using the Lemma 4.7 and the construction, we obtain the following bounds:

$$|r^i \partial_r^i \zeta_{big}| \lesssim |\hat{b} - b| \frac{\mathbf{1}_{r \leq 2B_1}}{1+r^2}, \quad |r^i \partial_r^i \nabla \xi_{big}| \lesssim |\hat{b} - b| \frac{r}{1+r^2} \mathbf{1}_{r \leq 2B_1}. \quad (4.66)$$

In the same way, using the Poisson field of the radial profile T_1 , we have:

$$|r^i \partial_r^i \nabla \phi_{\zeta_{big}}| \lesssim |\hat{b} - b| \frac{r(1 + \log r)}{1+r^2} \mathbf{1}_{r \leq 2B_1}. \quad (4.67)$$

The bounds (4.66) and (4.67) yield the first line of (4.56). Now, we use the formula:

$$\mathcal{L}^{(1)} \Big|_{\eta}^{\varepsilon} = \nabla \cdot (Q \nabla \mathcal{M}^{(1)}(\varepsilon, \eta)) = \Delta \varepsilon + Q \varepsilon + Q \Delta \eta + \nabla Q \nabla \eta + \nabla \varepsilon \nabla \phi_Q$$

to estimate

$$\begin{aligned} \mathcal{L}^{(1)} \left(\tilde{T}_1, \tilde{S}_1 \right) &= \Delta \tilde{T}_1 + Q \tilde{T}_1 + Q \Delta \tilde{S}_1 + \nabla Q \nabla \tilde{S}_1 + \nabla \tilde{T}_1 \nabla \phi_Q \\ &= \chi_{B_1} \Lambda Q + O \left(\frac{\mathbf{1}_{B_1 \leq r \leq 2B_1}}{1+r^4} \right) = \Lambda Q + O \left(\frac{\mathbf{1}_{r \geq B_1}}{1+r^4} \right) \end{aligned}$$

Moreover,

$$\mathcal{L}^{(2)} \left(\tilde{T}_1, \tilde{S}_1 \right) = \Delta \tilde{S}_1 - \tilde{T}_1 = \chi_{B_1} \Lambda \phi_Q + O \left(\frac{\mathbf{1}_{B_1 \leq r \leq 2B_1}}{1+r^2} \right) = \Lambda \phi_Q + O \left(\frac{\mathbf{1}_{r \geq B_1}}{1+r^2} \right)$$

Thus,

$$\mathcal{L} \Xi_{big} = (b - \hat{b}) \Lambda Q + \mathcal{R}, \quad (4.68)$$

where

$$\mathcal{R} = (b - \hat{b}) \begin{vmatrix} O \left(\frac{\mathbf{1}_{r \geq B_1}}{1+r^4} \right) \\ O \left(\frac{\mathbf{1}_{r \geq B_1}}{1+r^2} \right) \end{vmatrix} \quad (4.69)$$

So,

$$\int \frac{|\mathcal{L}^{(1)}(\zeta_{big}, \xi_{big})|^2}{Q} + \int |\nabla \mathcal{L}^{(2)}(\zeta_{big}, \xi_{big})|^2 \lesssim \frac{b^2}{|\log b|^2}.$$

To conclude the proof of the bound (4.56), we use the following equality, for function $\mathbf{f} = (f, g)$ well localized:

$$\mathcal{L} \Lambda \mathbf{f} = 2 \mathcal{L} \mathbf{f} + \Lambda (\mathcal{L} \mathbf{f}) - \begin{vmatrix} \Lambda^{(1)} \mathbf{Q} (f + \Delta g) + \nabla g \nabla (\Lambda^{(1)} \mathbf{Q}) + \nabla f \nabla \phi_{\Lambda \mathbf{Q}} \\ 0 \end{vmatrix}. \quad (4.70)$$

We compute $\mathcal{L} \mathbf{f}_\lambda$ and we differentiate this relation at $\lambda = 1$ to obtain (4.70), which use together with (4.68) and (4.69) yield the following bound:

$$\mathcal{L} \Lambda \Xi_{big} = (b - \hat{b}) \begin{vmatrix} O \left(\frac{\mathbf{1}_{r \geq B_1}}{1+r^4} \right) \\ O \left(\frac{\mathbf{1}_{r \geq B_1}}{1+r^2} \right) \end{vmatrix} \quad (4.71)$$

The bound (4.47) of the Lemma 4.7 concludes the proof of (4.56).

To prove (4.57) and (4.58), we use the same strategy, using the following bounds, coming from the Proposition 3.2:

$$\begin{aligned} |r^i \nabla^i \zeta_{sm}| &\lesssim \hat{b} \frac{\mathbf{1}_{\frac{B_1}{2} \leq r \leq 3B_1}}{r^2} \\ &+ |b - \hat{b}| \left[r^2 \mathbf{1}_{r \leq 1} + \frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^2 r^4 |\log b|} \mathbf{1}_{6B_0 \leq r \leq 2B_1} \right], \end{aligned} \quad (4.72)$$

$$|\nabla\phi_{\zeta_{sm}}| \lesssim \frac{\hat{b}}{r} \mathbf{1}_{r \geq \frac{B_1}{2}} + b|b - \hat{b}| \left[r^5 \mathbf{1}_{r \leq 1} + \frac{1 + r|\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{rb|\log b|} \mathbf{1}_{r \geq 2B_0} \right], \quad (4.73)$$

and

$$\begin{aligned} |r^i \partial_r^i \nabla \xi_{sm}| &\lesssim \hat{b} \frac{r}{1+r^2} \mathbf{1}_{\frac{B_1}{2} \leq r \leq 3B_1} \\ &+ b|b - \hat{b}| \left[r^5 \mathbf{1}_{r \leq 1} + r \frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{rb|\log b|} \mathbf{1}_{6B_0 \leq r \leq 2B_1} \right]. \end{aligned} \quad (4.74)$$

The whole proof is available in [36].

We prove now (4.60). The second part is a simple consequence of the above estimates. The first part is more technical, and we must use the structure of ζ . Indeed,

$$\zeta = (b - \hat{b})\tilde{T}_1 + \hat{b}(\chi_{B_1} - \chi_{\hat{B}_1})T_1 + \int_{\hat{b}}^b [2b\tilde{T}_2 + b^2\partial_b\tilde{T}_2] db$$

The crucial point here is the degeneracy of ΛT_1 :

$$\Lambda T_1 = O\left(\frac{\log r}{1+r^4}\right).$$

Hence, using the bound (4.72):

$$\begin{aligned} &\int (1 + \tau^2) |\Lambda \zeta|^2 \\ &\lesssim |b - \hat{b}|^2 \int_{\tau \leq 2B_1} \frac{|\log r|^2}{(1 + \tau^6)} + |b - \hat{b}|^2 \int_{B_1 \leq \tau \leq 2B_1} \frac{1}{1 + \tau^2} + |\hat{b}|^2 \int \frac{|\chi_{B_1} - \chi_{\hat{B}_1}|^2}{1 + \tau^2} \\ &+ |\hat{b}|^2 |b - \hat{b}|^2 \left\{ \int_{\tau \leq B_0} \tau^2 + \frac{1}{b^4 |\log b|^2} \int_{B_0 \leq \tau \leq 2B_1} \frac{1}{1 + \tau^4} \right\} \\ &\lesssim \frac{b^2}{|\log b|^2} + |\hat{b}|^2 \left| \log(\hat{B}_1) - \log(B_1) \right| \lesssim \frac{b^2}{|\log b|^2}. \end{aligned}$$

Using this bound together the following estimation

$$\left| \int Q |\nabla \mathcal{M}^{(1)} \Lambda \Xi|^2 \right| \lesssim \int (1 + \tau^2) |\Lambda \zeta|^2 + \int \frac{|\nabla \xi|^2}{1 + \tau^4},$$

and (4.56) and (4.57) concludes the proof of (4.60).

Now, let $\mathbf{v} = (v, v_1) \in L_Q^2 \times L^2$ with $\int v = 0$.

$$\begin{aligned} |\langle \mathcal{M} \Xi_2, \mathbf{v} \rangle| &\lesssim |\langle \mathcal{M} \mathcal{L} \Xi_{big}, \mathbf{v} \rangle| + |\langle \mathcal{M} \mathcal{L} \Xi_{sm}, \mathbf{v} \rangle| \\ &\lesssim |(b - \hat{b}) \langle \mathcal{M} \Lambda \mathbf{Q}, \mathbf{v} \rangle| + |\langle \mathcal{M} \mathcal{R}, \mathbf{v} \rangle| + |\langle \mathcal{M} \mathcal{L} \Xi_{sm}, \mathbf{v} \rangle| \end{aligned}$$

Using (2.6) and $\int v = 0$, we have :

$$\langle \mathcal{M} \Lambda \mathbf{Q}, \mathbf{v} \rangle = \int 2v = 0.$$

Now, from the estimate (4.69) :

$$|\nabla \phi_{\mathcal{R}^{(1)}}| = \left| \frac{1}{r} \int_0^r \mathcal{R}^{(1)}(\tau) \tau d\tau \right| \lesssim \frac{|b - \hat{b}|}{r} \int_0^r \frac{\mathbf{1}_{\tau \geq B_1} d\tau}{1 + \tau^3} \lesssim \frac{b^2}{r |\log b|^3}.$$

We estimate using the definition of the operator \mathcal{M} (2.2) :

$$\begin{aligned}
|\langle \mathcal{M}\mathcal{R}, \mathbf{v} \rangle| &\lesssim \left| \int v_1 \left(\frac{\mathcal{R}^{(1)}}{Q} + \mathcal{R}^{(2)} \right) \right| + \int \nabla v_2 \nabla \left(\mathcal{R}^{(2)} - \phi_{\mathcal{R}^{(1)}} \right) \\
&\lesssim \left[\int \frac{v_1^2}{Q} \int \frac{|b - \hat{b}|^2 \mathbf{1}_{\tau \geq B_1}}{Q(\tau)(1 + \tau^8)} \right]^{\frac{1}{2}} + \left[\int |\nabla v_2|^2 \left\{ \int |\nabla \phi_{\mathcal{R}^{(1)}}|^2 + \int |\nabla \mathcal{R}^{(2)}|^2 \right\} \right]^{\frac{1}{2}} \\
&\lesssim \frac{b^{\frac{3}{2}}}{|\log b|} \|\mathbf{v}\|_{X_Q}
\end{aligned}$$

To conclude, we estimate the term depending on Ξ_{sm} .

$$\begin{aligned}
|\langle \mathcal{M}\mathcal{L}\Xi_{sm}, \mathbf{v} \rangle| &= \left| \int \mathcal{M}^{(1)} \mathcal{L}\Xi_{sm} v_1 + \int \nabla \mathcal{M}^{(2)} \mathcal{L}\Xi_{sm} \nabla v_2 \right| \\
&\lesssim \left[\int Q \left| \mathcal{M}^{(1)} \mathcal{L}\Xi_{sm} \right|^2 \int \frac{v_1^2}{Q} \right]^{\frac{1}{2}} + \left[\int |\nabla v_2|^2 \left| \nabla \mathcal{M}^{(2)} \mathcal{L}\Xi_{sm} \right|^2 \right]^{\frac{1}{2}} \\
&\lesssim \|\mathcal{L}\Xi_{sm}\|_{X_Q} \|\mathbf{v}\|_{X_Q}.
\end{aligned}$$

The last inequality comes from the continuity of the operator \mathcal{M} (2.4). The bound (4.57) and the collection of above estimates yield (4.62). Using the same strategy, we prove (4.61). The proof is left to the reader.

To prove (4.64), we use the decomposition (4.68) and the knowledge of the kernel of \mathcal{L} .

$$\langle \mathcal{M}\Xi_2, \Xi_2 \rangle = \left\langle \mathcal{M}\Xi_2, (b - \hat{b})\Lambda Q + \mathcal{R} + \mathcal{L}\Xi_{sm} \right\rangle.$$

From the effect of the operator \mathcal{M} on the direction ΛQ , and that $\int \zeta_2 = 0$,

$$\left\langle \mathcal{M}\Xi_2, (b - \hat{b})\Lambda Q \right\rangle = 0.$$

Now from (4.62), (4.57) and (4.69)

$$\begin{aligned}
|\langle \mathcal{M}\Xi_2, \mathcal{R} + \mathcal{L}\Xi_{sm} \rangle| &\lesssim \frac{b^{\frac{3}{2}}}{|\log b|} [\|\mathcal{R}\|_{X_Q} + \|\mathcal{L}\Xi_{sm}\|_{X_Q}] \\
&\lesssim \frac{b^{\frac{3}{2}}}{|\log b|} \left[\left(\frac{b^2}{|\log b|^2} \int \frac{\mathbf{1}_{\tau \geq B_1}}{1 + \tau^4} \right)^{\frac{1}{2}} + \frac{b^{\frac{3}{2}}}{|\log b|} \right] \lesssim \frac{b^3}{|\log b|^2}.
\end{aligned}$$

This concludes the proof of (4.64). We prove now the last inequality (4.65). In this purpose, we can remark that :

$$\begin{aligned}
\begin{vmatrix} \nabla \mathcal{M}^{(1)} \\ \nabla \mathcal{M}^{(2)} \end{vmatrix} (\Xi_2) &= \begin{vmatrix} \nabla \mathcal{M}^{(1)} \\ \nabla \mathcal{M}^{(2)} \end{vmatrix} \left((b - \hat{b})\Lambda Q + \mathcal{R} + \mathcal{L}\Xi_{sm} \right) \\
&= \begin{vmatrix} \nabla \mathcal{M}^{(1)} \\ \nabla \mathcal{M}^{(2)} \end{vmatrix} (\mathcal{R} + \mathcal{L}\Xi_{sm})
\end{aligned}$$

Using the bounds (4.69), (4.72), (4.73) and (4.74), we obtain:

$$\begin{aligned}
\int Q \left| \nabla \mathcal{M}^{(1)} \mathcal{R} \right|^2 &\lesssim \frac{b^4}{|\log b|^6} \\
\int \left| \mathcal{L}^{(2)} \mathcal{R} \right|^2 &\lesssim \frac{b^5}{|\log b|^8} \\
\int Q \left| \nabla \mathcal{M}^{(1)} \mathcal{L}\Xi_{sm} \right|^2 + \int \left| \mathcal{L}^{(2)} \mathcal{L}\Xi_{sm} \right|^2 &\lesssim \frac{b^4}{|\log b|^2}
\end{aligned}$$

The last inequality comes from:

$$\int Q \left| \nabla \mathcal{M}^{(1)} \mathcal{L} \tilde{\mathbf{T}}_2 \right|^2 + \int \left| \mathcal{L}^{(2)} \mathcal{L} \tilde{\mathbf{T}}_2 \right|^2 \lesssim 1$$

and, $\forall i \geq 0$

$$\left| r^i \partial_r^i \mathcal{L} \Xi_{sm} \right| \lesssim b |b - \hat{b}| |\tilde{\mathbf{T}}_2| \lesssim \frac{b^2}{|\log b|} |\tilde{\mathbf{T}}_2|$$

(4.65) is proved and the Lemma 4.8 too. \square

4.6. Sharp modulation equation. We are in position to compute the sharp modulation equation. The lifted parameter \hat{b} plays here a crucial rule.

Lemma 4.9 (Sharp modulation equations for b). *There exist $C(M)$ an universal enough large constant, independant of $K^*(M)$, such that:*

$$\left| \hat{b}_s + \frac{2b^2}{|\log b|} \right| \lesssim C(M) \frac{b^2}{|\log b|^2}. \quad (4.75)$$

Proof. Step 1 Projection of the equation (4.27) satisfied by \mathbf{E} onto $\mathcal{L}^* \Phi_{0, \hat{B}_0}$.

We take the scalar product of (4.27) with $\mathcal{L}^* \Phi_{0, \hat{B}_0}$ and we reorganize the terms :

$$\begin{aligned} \partial_s \left\{ \left\langle \tilde{\mathbf{Q}}_b - \mathbf{Q} + \mathbf{E}, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \right\} &= \left\langle \tilde{\mathbf{Q}}_b - \mathbf{Q} + \mathbf{E}, \partial_s \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle + \left\langle \mathbf{E}_2, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \\ &+ \left\langle -\frac{\lambda_s}{\lambda} \Lambda \mathbf{E} + \mathbf{F} + \mathbf{G}, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \\ &+ \left(b + \frac{\lambda_s}{\lambda} \right) \left\langle \Lambda \tilde{\mathbf{Q}}_b, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle, \end{aligned} \quad (4.76)$$

where we recall that

$$\mathbf{F} = \tilde{\Psi}_b + \Theta_b(\varepsilon, \eta) + \mathbf{N}(\varepsilon, \eta)$$

with $\tilde{\Psi}_b$ is defined by (3.116) and

$$\begin{aligned} \Theta_b(\varepsilon, \eta) &= \begin{pmatrix} \nabla \cdot (\varepsilon \nabla \tilde{\gamma}_b + \tilde{\alpha}_b \nabla \eta) \\ 0 \end{pmatrix}, \\ \mathbf{N}(\varepsilon, \eta) &= \begin{pmatrix} \nabla \cdot (\varepsilon \nabla \eta) \\ 0 \end{pmatrix} \end{aligned}$$

and

$$\mathbf{G} = -c_b b^2 \check{\mathbf{T}}.$$

Step 2 Crucial rule of the second decomposition.

We use here the second decomposition to execute the error term. Indeed,

$$\left\langle \tilde{\mathbf{Q}}_b - \mathbf{Q} + \mathbf{E}, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle = \left\langle \tilde{\mathbf{Q}}_{\hat{b}} - \mathbf{Q} + \hat{\mathbf{E}}, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle = \left\langle \tilde{\mathbf{Q}}_{\hat{b}} - \mathbf{Q}, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle,$$

the last cancellation coming from the orthogonality condition (4.46). Hence, we can split the left term of (4.76) in two parts:

$$\partial_s \left\{ \left\langle \tilde{\mathbf{Q}}_b - \mathbf{Q} + \mathbf{E}, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \right\} = \left\langle \partial_s \tilde{\mathbf{Q}}_{\hat{b}}, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle + \left\langle \tilde{\mathbf{Q}}_{\hat{b}} - \mathbf{Q}, \partial_s \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \quad (4.77)$$

We estimate both terms separately. First,

$$\begin{aligned} \left\langle \partial_s \tilde{\mathbf{Q}}_{\hat{b}}, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle &= \hat{b}_s \left\langle \tilde{\mathbf{T}}_1 + \hat{b} \partial_b \tilde{\mathbf{T}}_1 + 2\hat{b} \tilde{\mathbf{T}}_2 + \hat{b}^2 \partial_b \tilde{\mathbf{T}}_2, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \\ &= \hat{b}_s \left\langle \mathbf{T}_1, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle + \hat{b}_s \left\langle \left(\tilde{\mathbf{T}}_1 - \mathbf{T}_1 \right) + \hat{b} \partial_b \tilde{\mathbf{T}}_1 + 2\hat{b} \tilde{\mathbf{T}}_2 + \hat{b}^2 \partial_b \tilde{\mathbf{T}}_2, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \end{aligned} \quad (4.78)$$

Using the bound (2.39) and the bounds of the Proposition 3.2, we obtain :

$$\begin{aligned}
& \left| \left\langle \left(\tilde{\mathbf{T}}_1 - \mathbf{T}_1 \right) + \hat{b} \partial_b \tilde{\mathbf{T}}_1 + 2\hat{b} \tilde{\mathbf{T}}_2 + \hat{b}^2 \partial_b \tilde{\mathbf{T}}_2, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \right| \\
& \lesssim \int_{\tau \leq \hat{B}_0} \left| \tilde{T}_1 - T_1 \right| + \hat{b} |\partial_b \tilde{T}_1| + 2\hat{b} |\tilde{T}_2| + \hat{b}^2 |\partial_b \tilde{T}_2| \\
& + \int_{\hat{B}_0 \leq \tau \leq 2\hat{B}_0} \frac{|\nabla \tilde{S}_1 - \nabla S_1| + \hat{b} |\partial_b \nabla \tilde{S}_1| + 2\hat{b} |\nabla \tilde{S}_2| + \hat{b}^2 |\partial_b \nabla \tilde{S}_2|}{1 + \tau^3} \\
& \lesssim \int_{B_1 \leq \tau \leq 2B_1} \frac{1}{\tau^2} + b \int \left[\frac{1 + |\log(\tau \sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq \tau \leq 6B_0} + \frac{1}{b^2 \tau^4 |\log b|} \mathbf{1}_{6B_0 \leq \tau \leq 2B_1} \right] \\
& + \int_{B_1 \leq \tau \leq 2B_1} \frac{1}{1 + \tau^4} + b \int_{1\tau 2B_1} \frac{\tau \log \tau}{1 + \tau^3} \\
& \lesssim 1
\end{aligned}$$

Now, the estimate (2.37) yields

$$\left\langle \mathbf{T}_1, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle = \left\langle \Lambda \mathbf{Q}, \Phi_{0, \hat{B}_0} \right\rangle = \left[-32\pi \log \hat{B}_0 + O(1) \right] \quad (4.79)$$

Hence, this estimate together (4.78) and (4.79) yields

$$\left\langle \partial_s \tilde{\mathbf{Q}}_{\hat{b}}, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle = \hat{b}_s \left[-32\pi \log \hat{B}_0 + O(1) \right] \quad (4.80)$$

For the second term of (4.77), we use the definition of Φ_{0, \hat{B}_0} (2.35), and the bounds of the Proposition 3.2:

$$\begin{aligned}
& \left| \left\langle \tilde{\mathbf{Q}}_{\hat{b}} - \mathbf{Q}, \partial_s \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \right| = \left| \left\langle \mathcal{L} \left(\tilde{\mathbf{Q}}_{\hat{b}} - \mathbf{Q} \right), \partial_s \Phi_{0, \hat{B}_0} \right\rangle \right| \\
& \lesssim |\hat{b}_s| \left[\int_{\hat{B}_0 \leq \tau \leq 2\hat{B}_0} \tau^2 \mathcal{L}^{(1)} \left(\tilde{\mathbf{Q}}_{\hat{b}} - \mathbf{Q} \right) + \int_{\hat{B}_0 \leq \tau \leq 2\hat{B}_0} \frac{\log \tau}{\tau} \nabla \mathcal{L}^{(2)} \left(\tilde{\mathbf{Q}}_{\hat{b}} - \mathbf{Q} \right) \right] \\
& \lesssim |\hat{b}_s| \left[\int_{\frac{B_0}{4} \leq \tau \leq 4B_0} \frac{\tau^2}{\tau^4} + \int_{\frac{B_0}{4} \leq \tau \leq 4B_0} \frac{\log \tau}{\tau^4} \right] \lesssim |\hat{b}_s| \quad (4.81)
\end{aligned}$$

Hence, using the above bound with (4.80), we obtain :

$$\partial_s \left\{ \left\langle \tilde{\mathbf{Q}}_{\hat{b}} - \mathbf{Q} + \mathbf{E}, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \right\} = \hat{b}_s \left[-32\pi \log \hat{B}_0 + O(1) \right] \quad (4.82)$$

Step 3 Estimations of the RHS of (4.76)

The leading term of the RHS of (4.76) is \mathbf{G} defined by

$$\mathbf{G} = -c_b b^2 \check{\mathbf{T}}.$$

By definition of $\check{\mathbf{T}}$ (3.26), we have

$$\left\langle \mathbf{G}, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle = -c_b b^2 \left[\left\langle \mathcal{L} \mathbf{T}_1, \Phi_{0, \hat{B}_0} \right\rangle + \left\langle (\chi_{\frac{B_0}{4}} - 1) \mathbf{T}_1, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \right]$$

Now, from (2.39), we have

$$\left| \left\langle (\chi_{\frac{B_0}{4}} - 1) \mathbf{T}_1, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \right| \lesssim \left| \int_{r \leq 2\hat{B}_0} \frac{\mathbf{1}_{r \geq \frac{B_0}{4}}}{1 + r^2} + \log \hat{B}_0 \int_{\hat{B}_0 \leq \tau \leq 2\hat{B}_0} \frac{1}{1 + \tau^4} \right| \lesssim 1$$

The estimates (2.35) and the definition of c_b (3.50) yields:

$$\begin{aligned} \left\langle \mathbf{G}, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle &= \frac{2b^2}{|\log b|} \left[1 + O\left(\frac{1}{|\log b|}\right) \right] [32\pi \log B_0 + O(1)] \\ &= 32\pi b^2 \left[1 + O\left(\frac{1}{|\log b|}\right) \right]. \end{aligned}$$

We estimate like for the proof of (4.81) :

$$\left| \left\langle \tilde{\mathbf{Q}}_b - \mathbf{Q}, \partial_s \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \right| \lesssim |\hat{b}_s|,$$

and the linear term in \mathbf{E} with the bootstrap bound (4.16) :

$$\begin{aligned} \left| \left\langle \mathbf{E}, \partial_s \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \right| &= \left| \left\langle \mathbf{E}_2, \partial_s \Phi_{0, \hat{B}_0} \right\rangle \right| \\ &\lesssim \left| \frac{(\hat{B}_0)_s}{\hat{B}_0} \right| \left| \int \varepsilon_2 \left(r \chi' \frac{\tau}{\hat{B}_0} \right) \tau^2 + \int \nabla \eta_2 \left(r \chi' \frac{\tau}{\hat{B}_0} \right) \frac{\log(1 + \tau^2)}{1 + \tau} \right| \\ &\lesssim \frac{\hat{b}_s}{b} \|\mathbf{E}_2\|_{X_Q} \left(\int_{\hat{B}_0 \leq \tau \leq 2\hat{B}_0} \frac{\tau^4}{\tau^4} + \int_{\hat{B}_0 \leq \tau \leq 2\hat{B}_0} \frac{|\log \tau|^2}{\tau^2} \right)^{\frac{1}{2}} \\ &\lesssim \frac{\hat{b}_s}{b\sqrt{b}} \|\mathbf{E}_2\|_{X_Q} \lesssim |\hat{b}_s|. \end{aligned}$$

Now, we focus on the main liner term, using the bound (2.41) :

$$\left| \left\langle \mathbf{E}_2, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \right| \lesssim \int_{\tau \geq \hat{B}_0} |\varepsilon_2| + \log \hat{B}_0 \int_{\hat{B}_0 \leq \tau \leq 2\hat{B}_0} \frac{|\nabla \eta_2|}{1 + \tau^3} \lesssim \sqrt{b} \|\mathbf{E}_2\|_{X_Q} \lesssim \frac{b^2}{|\log b|}.$$

Next, with the bound (2.39), the bootstrap bound (4.16), the interpolation bound (C.1) and the bound (4.38) coming from the rough modulation equation :

$$\begin{aligned} \left| \left\langle \frac{\lambda_s}{\lambda} \Lambda \mathbf{E}, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \right| &\lesssim b \left\{ \int_{\tau \leq 2\hat{B}_0} (|\varepsilon| + |\tau \cdot \nabla \varepsilon|) + \int_{\hat{B}_0 \leq \tau \leq 2\hat{B}_0} \frac{|\nabla \eta| + |\tau \cdot \nabla^2 \eta|}{1 + \tau^3} \right\} \\ &\lesssim \sqrt{b} (\|\varepsilon\|_{L^2} + \|y \cdot \nabla \varepsilon\|_{L^2} + \left\| \frac{\nabla \eta}{1 + y^3} \right\|_{L^2} + \left\| \frac{\nabla^2 \eta}{1 + y^2} \right\|_{L^2}) \\ &\lesssim C(M) \sqrt{b} \|\mathbf{E}_2\|_{X_Q} \lesssim \frac{b^2}{|\log b|} \end{aligned}$$

We treat the \mathbf{F} terms separately. To begin, from the Proposition 3.2, we know already the degenerate flux :

$$\left| \left\langle \tilde{\Psi}_b, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \right| \lesssim \frac{b^2}{|\log b|}.$$

Let's focus on the small linear term $\Theta_b(\varepsilon, \eta)$. We recall that

$$\Theta_b(\varepsilon, \eta) = \begin{vmatrix} \nabla \cdot (\varepsilon \nabla \tilde{\gamma}_b + \tilde{\alpha}_b \nabla \eta) \\ 0 \end{vmatrix}$$

and from the Proposition 3.2, we have the rough bounds, $\forall i \geq 0$

$$\begin{aligned} |r^i \partial_r^i \tilde{\alpha}_b| &\lesssim \frac{b}{1 + r^2} \mathbf{1}_{r \leq 2B_1} \\ |r^i \partial_r^i \nabla \tilde{\gamma}_b| &\lesssim \frac{br(1 + |\log r|)}{1 + r^2} \mathbf{1}_{r \leq 2B_1} \end{aligned}$$

Hence, we compute the following rough bound :

$$\left| \Theta_b^{(1)}(\varepsilon, \eta) \right| \lesssim b \left[\frac{1 + |\log r|}{1 + r} |\nabla \varepsilon| + \frac{1 + \log r}{1 + r^2} |\varepsilon| + \frac{|\Delta \eta|}{1 + r^2} + \frac{|\nabla \eta|}{1 + r^3} \right]$$

The estimate (2.41) and the cancellation $\int \Theta_b^{(1)} = 0$ yields:

$$\begin{aligned} \left| \left\langle \Theta_b(\varepsilon, \eta), \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \right| &\lesssim b \int_{r \geq \hat{B}_0} \left[\frac{1 + |\log r|}{1 + r} |\nabla \varepsilon| + \frac{1 + \log r}{1 + r^2} |\varepsilon| + \frac{|\Delta \eta|}{1 + r^2} + \frac{|\nabla \eta|}{1 + r^3} \right] \\ &\lesssim C(M) \sqrt{b} \|\mathbf{E}_2\|_{X_Q} \lesssim \frac{b^2}{|\log b|}. \end{aligned}$$

To conclude, we must treat the nonlinear term defined by

$$\mathbf{N}(\varepsilon, \eta) = \begin{vmatrix} \nabla \cdot (\varepsilon \nabla \eta) \\ 0 \end{vmatrix}$$

As $\int N^{(1)}(\varepsilon, \eta) = 0$, we can use the same strategy like the small linear term :

$$\begin{aligned} \left| \left\langle \mathbf{N}(\varepsilon, \eta), \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \right| &\lesssim b \int_{r \geq \hat{B}_0} |\nabla \cdot (\varepsilon \nabla \eta)| \\ &\lesssim \left(\int |\varepsilon|^2 \int |\Delta \eta|^2 \right)^{\frac{1}{2}} + \left(\int |\nabla \varepsilon|^2 \int |\nabla \eta|^2 \right)^{\frac{1}{2}} \end{aligned}$$

Now, we have

$$\int |\Delta \eta|^2 = \left| \int \nabla \eta \nabla \Delta \eta \right| \lesssim \left(\int |\nabla \Delta \eta|^2 \int |\nabla \eta|^2 \right)^{\frac{1}{2}}$$

The bootstrap bounds (4.13), (4.16) and the interpolation bound (C.1) yield

$$\left| \left\langle \mathbf{N}(\varepsilon, \eta), \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \right| \lesssim \frac{b^{\frac{3}{2}}}{|\log b|} b |\log b|^C \lesssim \frac{b^2}{|\log b|}$$

The collection of above estimates yield

$$\left| \left\langle \mathbf{F}, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \right| \lesssim \frac{b^2}{|\log b|}$$

For the last term induced by the modulation, we use the same strategy because of $\int \Lambda \tilde{Q}_b^{(1)} = \int \Lambda \tilde{\alpha}_b = 0$. Thus, from the bound (4.38) coming from the rough modulation equations,

$$\begin{aligned} \left| b + \frac{\lambda_s}{\lambda} \right| \left| \left\langle \Lambda \tilde{Q}_b, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \right| &= \left| b + \frac{\lambda_s}{\lambda} \right| \left| \left\langle \Lambda \tilde{\alpha}_b, \mathcal{L}^* \Phi_{0, \hat{B}_0} \right\rangle \right| \\ &\lesssim b^{\frac{3}{2}} \int_{\frac{B_0}{4} \leq r \leq 6B_0} \frac{b}{1 + r^2} \lesssim \frac{b^2}{|\log b|}. \end{aligned}$$

Injecting all above estimates into (4.76) yields :

$$\hat{b}_s[-32\pi \log \hat{B}_0 + O(1)] = 32\pi b^2 + O\left(\frac{b^2}{|\log b|} + |\hat{b}_s|\right).$$

Now, by assumption

$$\left| \frac{\log \hat{B}_0}{|\log b|} - \frac{1}{2} \right| = \left| \frac{\log(\hat{B}_0 \sqrt{b})}{|\log b|} \right| \lesssim \frac{1}{|\log b|}, \quad (4.83)$$

Hence,

$$\left| \hat{b}_s + \frac{2b^2}{|\log b|} \right| \lesssim C(M) \frac{b^2}{|\log b|^2}.$$

This is precisely (4.75), and the Lemma 4.9 is proved. \square

5. X_Q monotonicity

The goal of this section is to prove the following monotonicity formula at the X_Q level, which is the keystone of the proof of (4.22).

Proposition 5.1 (X_Q monotonicity). *There holds:*

$$\begin{aligned} & \frac{d}{dt} \left\{ \frac{\langle \mathcal{M}\mathbf{E}_2, \mathbf{E}_2 \rangle}{\lambda^2} + O \left(\frac{C(M)b^{\frac{3}{2}}}{|\log b|} \|\mathbf{E}_2\|_{X_Q} + \frac{C(M)b^3}{\lambda^2 |\log b|^2} \right) \right\} \\ & \leq \frac{bC(M)}{\lambda^4} \left[\frac{b^{\frac{3}{2}}}{|\log b|} \|\mathbf{E}_2\|_{X_Q} + \frac{b^3}{|\log b|^2} \right]. \end{aligned} \quad (5.1)$$

Proof. To prove this proposition, which is the most technical step of the proof of the Theorem 1.1, we shall use the second decomposition of the flow. Indeed, we have seen that the control of b_s (4.37) is not enough good for our analysis, due to a too slow decay for the elements of the kernel of \mathcal{L}^* . To circumvent this technical problem, we have introduced the lift parameter \hat{b} , whose the control (4.75) of the time derivate is better. In the first step, we shall begin to write the equations verify by the error term coming from the second decomposition, and its suitable derivatives. Thus, we shall compute the modified energy identity, whose we control each term in the last step of the proof.

5.1. Equations verify by $\hat{\mathbf{E}}$ and its suitable derivatives: We recall the second decomposition of the flow.

$$\mathbf{U} = \begin{pmatrix} u \\ v \end{pmatrix} = \left(\tilde{\mathbf{Q}}_{\hat{b}} + \hat{\mathbf{E}} \right)_\lambda = \left(\begin{array}{c} Q + \tilde{\alpha}_{\hat{b}} + \hat{\varepsilon} \\ \phi_Q + \tilde{\gamma}_{\hat{b}} + \hat{\eta} \end{array} \right)_\lambda. \quad (5.2)$$

Moreover, we have defined

$$\Xi = \hat{\mathbf{E}} - \mathbf{E} = \begin{pmatrix} \zeta \\ \xi \end{pmatrix} \quad (5.3)$$

the gap between both decomposition. By definition:

$$\partial_s \left(\tilde{\mathbf{Q}}_{\hat{b}} + \hat{\mathbf{E}} \right) = \partial_s \left(\tilde{\mathbf{Q}}_b + \mathbf{E} \right)$$

Hence the equation (4.27) becomes:

$$\partial_s \hat{\mathbf{E}} - \frac{\lambda_s}{\lambda} \Lambda \hat{\mathbf{E}} = \mathcal{L} \hat{\mathbf{E}} + \mathbf{F} + \widehat{\mathbf{Mod}} + \mathbf{G} + \mathbf{H} = \mathcal{L} \hat{\mathbf{E}} + \hat{\mathcal{F}}, \quad (5.4)$$

where \mathbf{F} and \mathbf{G} are respectively defined by (4.28) and (4.32), and the new modulation term is given by

$$\widehat{\mathbf{Mod}}(s) = \left(b + \frac{\lambda_s}{\lambda} \right) \Lambda \tilde{\mathbf{Q}}_b - \partial_s \tilde{\mathbf{Q}}_{\hat{b}}, \quad (5.5)$$

and

$$\mathbf{H} = -\mathcal{L} \Xi - \frac{\lambda_s}{\lambda} \Lambda \Xi. \quad (5.6)$$

The decomposition of the flow in the original variables is given by

$$\mathbf{U} = \left(\tilde{\mathbf{Q}}_b \right)_\lambda + \widehat{\mathbf{W}} = \left(\begin{array}{c} Q + \tilde{\alpha}_b \\ \phi_Q + \tilde{\gamma}_b \end{array} \right)_\lambda + \begin{vmatrix} \hat{w} \\ \hat{z} \end{vmatrix}. \quad (5.7)$$

Hence, $\widehat{\mathbf{W}}$ satisfies the equation:

$$\partial_t \widehat{\mathbf{W}} = \mathcal{L}_\lambda \widehat{\mathbf{W}} + \frac{1}{\lambda^2} \widehat{\mathcal{F}}_\lambda \quad (5.8)$$

In the rest of the paper, we use the following notation, in order to ease the clarity of the calculus.

$$\nabla \cdot \mathbf{U} = \nabla \cdot \begin{vmatrix} u \\ v \end{vmatrix} = \begin{vmatrix} \nabla \cdot u \\ \nabla \cdot v \end{vmatrix} \quad (5.9)$$

Introduce the differential operator of order one, which appear in the factorization of the operator \mathcal{L} :

$$\mathcal{A}_\lambda \mathbf{W} = \begin{vmatrix} Q_\lambda \nabla \mathcal{M}_\lambda^{(1)}(w, z) \\ \nabla z - \nabla \phi_w \end{vmatrix} = \begin{vmatrix} \nabla w + \nabla \phi_{Q_\lambda} w + Q_\lambda \nabla z \\ \nabla z - \nabla \phi_w \end{vmatrix}. \quad (5.10)$$

Indeed, we have this relation between the operators \mathcal{A} and \mathcal{L} :

$$\mathcal{L}_\lambda \mathbf{W} = \nabla \cdot (\mathcal{A}_\lambda \mathbf{W}) \quad (5.11)$$

In the following, we use the following notations for the suitable derivatives of order one:

$$\begin{aligned} \mathbf{W}_1 &= \begin{vmatrix} w_1 \\ z_1 \end{vmatrix} = A_\lambda \mathbf{W}, & \widehat{\mathbf{W}}_1 &= \begin{vmatrix} \hat{w}_1 \\ \hat{z}_1 \end{vmatrix} = A_\lambda \widehat{\mathbf{W}}, \\ \mathbf{E}_1 &= \begin{vmatrix} \varepsilon_1 \\ \eta_1 \end{vmatrix} = A \mathbf{E}, & \widehat{\mathbf{E}}_1 &= \begin{vmatrix} \hat{\varepsilon}_1 \\ \hat{\eta}_1 \end{vmatrix} = A \widehat{\mathbf{E}}, \end{aligned}$$

and these notations for the suitable derivatives of order two:

$$\begin{aligned} \mathbf{W}_2 &= \begin{vmatrix} w_2 \\ z_2 \end{vmatrix} = \mathcal{L}_\lambda \mathbf{W}, & \widehat{\mathbf{W}}_2 &= \begin{vmatrix} \hat{w}_2 \\ \hat{z}_2 \end{vmatrix} = \mathcal{L}_\lambda \widehat{\mathbf{W}} \\ \mathbf{E}_2 &= \begin{vmatrix} \varepsilon_2 \\ \eta_2 \end{vmatrix} = \mathcal{L} \mathbf{E}, & \widehat{\mathbf{E}}_2 &= \begin{vmatrix} \hat{\varepsilon}_2 \\ \hat{\eta}_2 \end{vmatrix} = \mathcal{L} \widehat{\mathbf{E}} \end{aligned}$$

In the same way, we notice:

$$\mathbf{\Xi}_1 = \begin{vmatrix} \zeta_1 \\ \xi_1 \end{vmatrix} = A \mathbf{\Xi}, \quad \mathbf{\Xi}_2 = \begin{vmatrix} \zeta_2 \\ \xi_2 \end{vmatrix} = \mathcal{L} \mathbf{\Xi} \quad (5.12)$$

Using these new notations, (5.8) becomes:

$$\partial_t \widehat{\mathbf{W}} = \widehat{\mathbf{W}}_2 + \frac{1}{\lambda^2} \widehat{\mathcal{F}}_\lambda = \widehat{\mathbf{W}}_2 + \frac{1}{\lambda^2} \begin{vmatrix} \widehat{\mathcal{F}}_{1,\lambda} \\ \widehat{\mathcal{F}}_{2,\lambda} \end{vmatrix}. \quad (5.13)$$

Now, we introduce a second operator of order one

$$V_\lambda \widehat{\mathbf{W}} = [\partial_t, A_\lambda] \widehat{\mathbf{W}} = \begin{vmatrix} -\partial_t (\nabla \phi_{Q_\lambda}) \hat{w} - \partial_t Q_\lambda \nabla \hat{z} \\ 0 \end{vmatrix} = \frac{\lambda_s}{\lambda} \begin{vmatrix} (\nabla \phi_{\Lambda Q})_\lambda \hat{w} + (\Lambda Q)_\lambda \nabla \hat{z} \\ 0 \end{vmatrix}. \quad (5.14)$$

Hence $\widehat{\mathbf{W}}_1$ and $\widehat{\mathbf{W}}_2$ are respectively solutions of:

$$\partial_t \widehat{\mathbf{W}}_1 = A_\lambda \widehat{\mathbf{W}}_2 + \frac{1}{\lambda^2} A_\lambda \widehat{\mathcal{F}}_\lambda + V_\lambda \widehat{\mathbf{W}}, \quad (5.15)$$

$$\partial_t \widehat{\mathbf{W}}_2 = \mathcal{L}_\lambda \widehat{\mathbf{W}}_2 + \frac{1}{\lambda^2} \mathcal{L}_\lambda \widehat{\mathcal{F}}_\lambda + \nabla \cdot (V_\lambda \widehat{\mathbf{W}}). \quad (5.16)$$

5.2. Modified energy identity.

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \langle \mathcal{M}_\lambda \widehat{\mathbf{W}}_2, \widehat{\mathbf{W}}_2 \rangle = \langle \partial_t \widehat{\mathbf{W}}_2, \mathcal{M}_\lambda \widehat{\mathbf{W}}_2 \rangle - \int \frac{\partial_t Q_\lambda}{2Q_\lambda^2} \hat{w}_2^2 \\
& = \left\langle \mathcal{L}_\lambda \widehat{\mathbf{W}}_2 + \frac{1}{\lambda^2} \mathcal{L}_\lambda \widehat{\mathcal{F}}_\lambda + \nabla \cdot (V_\lambda \widehat{\mathbf{W}}), \mathcal{M}_\lambda \widehat{\mathbf{W}}_2 \right\rangle - \int \frac{\partial_t Q_\lambda}{2Q_\lambda^2} \hat{w}_2^2 \\
& = - \int Q_\lambda |\nabla \mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2)|^2 - \int |\Delta \mathcal{M}_\lambda^{(2)}(\hat{w}_2, \hat{z}_2)|^2 \\
& + \left\langle \widehat{\mathbf{W}}_2, \frac{1}{\lambda^2} \mathcal{M}_\lambda \mathcal{L}_\lambda \widehat{\mathcal{F}}_\lambda \right\rangle - \langle V_\lambda \widehat{\mathbf{W}}, \nabla \mathcal{M}_\lambda \widehat{\mathbf{W}}_2 \rangle \\
& + \int \frac{\hat{w}_2^2}{2\lambda^2 Q_\lambda^2} \left[\left(\hat{b} + \frac{\lambda_s}{\lambda} \right) (\Lambda Q)_\lambda \right] - \int \frac{\hat{b}(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \hat{w}_2^2. \tag{5.17}
\end{aligned}$$

For our analysis, the last term has a critical size. Moreover, this term hasn't definite sign. Hence, we must decompose this term in several manageable terms. In this purpose, we compute :

$$\begin{aligned}
& \frac{d}{dt} \left\{ \int \frac{\hat{b}(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \hat{w}_2 \hat{w} \right\} \tag{5.18} \\
& = \int \frac{d}{dt} \left\{ \frac{\hat{b}(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \right\} \hat{w}_2 \hat{w} + \int \frac{\hat{b}(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \hat{w}_2 \left[\hat{w}_2 + \frac{1}{\lambda^2} \widehat{\mathcal{F}}_{1,\lambda} \right] \\
& + \int \frac{\hat{b}(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \hat{w} \left[\mathcal{L}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2) + \frac{1}{\lambda^2} \mathcal{L}_\lambda^{(1)}(\widehat{\mathcal{F}}_{1,\lambda}, \widehat{\mathcal{F}}_{2,\lambda}) + \nabla \cdot V_\lambda^{(1)}(\hat{w}, \hat{y}) \right].
\end{aligned}$$

But,

$$\begin{aligned}
& \frac{d}{dt} \left\{ \frac{\hat{b}(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \right\} = \frac{d}{dt} \left\{ \frac{\hat{b}}{2} \left(\frac{\Lambda Q}{Q^2} \right) \left(\frac{r}{\lambda(t)} \right) \right\} \tag{5.19} \\
& = \frac{1}{2\lambda^2} \left[\hat{b}_s \frac{\Lambda Q}{Q^2} - \hat{b} \frac{\lambda_s}{\lambda} y \cdot \nabla \left(\frac{\Lambda Q}{Q^2} \right) \right] (y) \\
& = \frac{1}{\lambda^2 Q} \left[4\hat{b} \frac{\lambda_s}{\lambda} - \hat{b}_s + O \left(\frac{|\hat{b}_s| + \hat{b} \left| \frac{\lambda_s}{\lambda} \right|}{1+r^2} \right) \right].
\end{aligned}$$

Now, using a intregation by part, we obtain

$$\begin{aligned}
& \int \frac{\hat{b}(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \hat{w} \mathcal{L}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2) = -\hat{b} \int \frac{\hat{w} \mathcal{L}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2)}{\lambda^2 Q_\lambda} + \int \frac{\hat{b}(2Q + \Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \hat{w} \mathcal{L}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2) \\
& = \frac{\hat{b}}{\lambda^2} \int Q_\lambda \nabla(\mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2)) \cdot \nabla \left[\frac{\hat{w}}{Q_\lambda} + \hat{z} \right] \\
& - \frac{\hat{b}}{2\lambda^2} \int Q_\lambda \nabla(\mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2)) \cdot \nabla \left[2\hat{z} + \frac{(2Q + \Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w} \right] \tag{5.20} \\
& = -\frac{\hat{b}}{\lambda^2} \int \hat{w}_2 \mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2) - \frac{\hat{b}}{2\lambda^2} \int Q_\lambda \nabla(\mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2)) \cdot \nabla \left[2\hat{z} + \frac{(2Q + \Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w} \right]
\end{aligned}$$

We have used in the above equality the fundamental degeneracy :

$$\frac{\Lambda Q}{Q} + 2 = -\phi_{\Lambda Q} = O \left(\frac{1}{1+r^2} \right). \tag{5.21}$$

Injecting (5.19) and (5.20) in (5.18) yield

$$\begin{aligned}
& \frac{d}{dt} \left\{ \int \frac{\hat{b}(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \hat{w}_2 \hat{w} \right\} = \int \frac{\hat{b}(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \hat{w}_2^2 - \frac{\hat{b}}{\lambda^2} \int \hat{w}_2 \mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2) \\
& + \int \frac{\hat{w}_2 \hat{w}}{\lambda^2 Q_\lambda} \left[4\hat{b} \frac{\lambda_s}{\lambda} - \hat{b}_s \right] - \frac{\hat{b}}{2\lambda^2} \int Q_\lambda \nabla(\mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2)) \cdot \nabla \left[2\hat{z} + \frac{(2Q + \Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w} \right] \\
& + \int \frac{\hat{b}(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \hat{w} \left[\frac{1}{\lambda^2} \mathcal{L}_\lambda^{(1)}(\hat{\mathcal{F}}_{1,\lambda}, \hat{\mathcal{F}}_{2,\lambda}) + \nabla \cdot V_\lambda^{(1)}(\hat{w}, \hat{y}) \right] + \int \frac{\hat{b}(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \hat{w}_2 \left[\frac{1}{\lambda^2} \hat{\mathcal{F}}_{1,\lambda} \right] \\
& + O \left(\int \frac{\hat{w}_2 \hat{w}}{\lambda^2 Q(1+r^2)} \left[|\hat{b}_s| + \hat{b} \left| \frac{\lambda_s}{\lambda} \right| \right] \right).
\end{aligned} \tag{5.22}$$

Summing this equality with (5.17), we obtain a first modified energy identity. However, some terms have still critical size for our analysis. To solve this problem, we use the energy identity of \hat{w}_1 :

$$\begin{aligned}
& -\frac{d}{dt} \left\{ \int \frac{\hat{b} \hat{w}_1^2}{\lambda^2 Q_\lambda} \right\} \\
& = -\int \hat{w}_1^2 \frac{d}{dt} \left\{ \frac{\hat{b}}{\lambda^2 Q_\lambda} \right\} - 2\hat{b} \int \frac{\hat{w}_1}{\lambda^2 Q_\lambda} \left[A_\lambda^{(1)}(\hat{w}_2, \hat{z}_2) + \frac{1}{\lambda^2} A_\lambda^{(1)}(\hat{\mathcal{F}}_{1,\lambda}, \hat{\mathcal{F}}_{2,\lambda}) + V_\lambda^{(1)}(\hat{w}, \hat{z}) \right]
\end{aligned} \tag{5.23}$$

Next, we compute:

$$\begin{aligned}
& -\frac{d}{dt} \left\{ \frac{\hat{b}}{\lambda^2 Q_\lambda} \right\} = -\frac{d}{dt} \left\{ \frac{\hat{b}}{Q \left(\frac{r}{\lambda(t)} \right)} \right\} = \frac{-1}{\lambda^2} \left[\frac{\hat{b}_s}{Q} + \hat{b} \frac{\lambda_s}{\lambda} \frac{y \cdot \nabla Q}{Q^2} \right] \\
& = \frac{-1}{\lambda^2 Q_\lambda} \left[4\hat{b} \frac{\lambda_s}{\lambda} - \hat{b}_s + O \left(\frac{|\hat{b}_s| + \hat{b} \left| \frac{\lambda_s}{\lambda} \right|}{1+r^2} \right) \right],
\end{aligned}$$

Moreover:

$$-2\hat{b} \int \frac{\hat{w}_1}{\lambda^2 Q_\lambda} A_\lambda^{(1)}(\hat{w}_2, \hat{z}_2) = 2\frac{\hat{b}}{\lambda^2} \int \nabla \cdot \hat{w}_1 \mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2) = 2\frac{\hat{b}}{\lambda^2} \int \hat{w}_2 \mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2)$$

Injecting both last terms in (5.23) yields:

$$\begin{aligned}
& -\frac{d}{dt} \left\{ \int \frac{\hat{b} \hat{w}_1^2}{\lambda^2 Q_\lambda} \right\} = 2\frac{\hat{b}}{\lambda^2} \int \hat{w}_2 \mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2) + \int \frac{\hat{w}_1^2}{\lambda^2 Q} \left[4\hat{b} \frac{\lambda_s}{\lambda} - \hat{b}_s \right] \\
& - 2\hat{b} \int \frac{\hat{w}_1}{\lambda^2 Q_\lambda} \left[\frac{1}{\lambda^2} A_\lambda^{(1)}(\hat{\mathcal{F}}_{1,\lambda}, \hat{\mathcal{F}}_{2,\lambda}) + V_\lambda^{(1)}(\hat{w}, \hat{z}) \right] + O \left(\int \frac{\hat{w}_1^2}{\lambda^2 Q(1+r^2)} \left[|\hat{b}_s| + \hat{b} \left| \frac{\lambda_s}{\lambda} \right| \right] \right).
\end{aligned} \tag{5.24}$$

Summing (5.22) with (5.24) yields:

$$\begin{aligned}
& \frac{d}{dt} \left\{ \int \frac{\hat{b}(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \hat{w}_2 \hat{w} - \int \frac{\hat{b} \hat{w}_1^2}{\lambda^2 Q_\lambda} \right\} = \int \frac{\hat{b}(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \hat{w}_2^2 + \frac{\hat{b}}{\lambda^2} \int \hat{w}_2 \mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2) \\
& + \int \frac{\hat{w}_2 \hat{w} + \hat{w}_1^2}{\lambda^2 Q_\lambda} \left[4\hat{b} \frac{\lambda_s}{\lambda} - \hat{b}_s \right] - \frac{\hat{b}}{2\lambda^2} \int Q_\lambda \nabla(\mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2)) \cdot \nabla \left[2\hat{z} + \frac{(2Q + \Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w} \right] \\
& + \int \frac{\hat{b}(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \hat{w} \left[\frac{1}{\lambda^2} \mathcal{L}_\lambda^{(1)}(\hat{\mathcal{F}}_{1,\lambda}, \hat{\mathcal{F}}_{2,\lambda}) + \nabla \cdot V_\lambda^{(1)}(\hat{w}, \hat{y}) \right] + \int \frac{\hat{b}(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \hat{w}_2 \left[\frac{1}{\lambda^2} \hat{\mathcal{F}}_{1,\lambda} \right] \\
& - 2\hat{b} \int \frac{\hat{w}_1}{\lambda^2 Q_\lambda} \left[\frac{1}{\lambda^2} A_\lambda^{(1)}(\hat{\mathcal{F}}_{1,\lambda}, \hat{\mathcal{F}}_{2,\lambda}) + V_\lambda^{(1)}(\hat{w}, \hat{z}) \right] \\
& + O \left(\int \frac{|\hat{w}_2 \hat{w}| + |\hat{w}_1|^2}{\lambda^2 Q(1+r^2)} \left[|\hat{b}_s| + \hat{b} \left| \frac{\lambda_s}{\lambda} \right| \right] \right). \tag{5.25}
\end{aligned}$$

By integration by parts, we have:

$$\int \frac{\hat{w}_2 \hat{w}}{Q_\lambda} = - \int Q_\lambda \nabla \mathcal{M}_\lambda^{(1)}(\hat{w}, \hat{z}) \cdot \nabla \left(\frac{\hat{w}}{Q_\lambda} \right) = - \int \frac{\hat{w}_1^2}{Q_\lambda} + \int Q_\lambda \nabla \mathcal{M}_\lambda^{(1)}(\hat{w}, \hat{z}) \cdot \nabla \hat{z}$$

and thus

$$\int \frac{\hat{w}_2 \hat{w} + \hat{w}_1^2}{Q_\lambda} = \int \hat{w}_1 \cdot \nabla \hat{z}. \tag{5.26}$$

In the same way:

$$\begin{aligned}
& - \int \frac{1}{Q_\lambda} \left\{ \hat{w} \mathcal{L}_\lambda^{(1)}(\hat{\mathcal{F}}_{1,\lambda}, \hat{\mathcal{F}}_{2,\lambda}) + \hat{w}_2 \hat{\mathcal{F}}_{1,\lambda} \right\} - 2 \int \frac{1}{Q_\lambda} \hat{w}_1 \cdot \mathcal{A}_\lambda^{(1)}(\hat{\mathcal{F}}_{1,\lambda}, \hat{\mathcal{F}}_{2,\lambda}) \\
& = \int Q_\lambda \nabla \mathcal{M}_\lambda^{(1)}(\hat{\mathcal{F}}_{1,\lambda}, \hat{\mathcal{F}}_{2,\lambda}) \cdot \nabla \left(\mathcal{M}_\lambda^{(1)}(\hat{w}, \hat{z}) - \hat{z} \right) \\
& + \int Q_\lambda \nabla \mathcal{M}_\lambda^{(1)}(\hat{w}, \hat{z}) \cdot \nabla \left(\mathcal{M}_\lambda^{(1)}(\hat{\mathcal{F}}_{1,\lambda}, \hat{\mathcal{F}}_{2,\lambda}) - \hat{\mathcal{F}}_{2,\lambda} \right) \\
& - 2 \int Q_\lambda \nabla \mathcal{M}_\lambda^{(1)}(\hat{\mathcal{F}}_{1,\lambda}, \hat{\mathcal{F}}_{2,\lambda}) \cdot \nabla \mathcal{M}_\lambda^{(1)}(\hat{w}, \hat{z}) \\
& = -2 \int \hat{w}_1 \cdot \nabla \hat{\mathcal{F}}_{2,\lambda} - 2 \int Q \nabla \mathcal{M}_\lambda^{(1)}(\hat{\mathcal{F}}_{1,\lambda}, \hat{\mathcal{F}}_{2,\lambda}) \cdot \nabla \hat{z}.
\end{aligned}$$

Hence:

$$\begin{aligned}
& \int \frac{\hat{b}(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \left[\frac{\hat{w}}{\lambda^2} \mathcal{L}_\lambda^{(1)}(\hat{\mathcal{F}}_{1,\lambda}, \hat{\mathcal{F}}_{2,\lambda}) + \frac{\hat{w}_2}{\lambda^2} \hat{\mathcal{F}}_{1,\lambda} \right] - 2\hat{b} \int \frac{\hat{w}_1}{\lambda^2 Q_\lambda} \left[\frac{1}{\lambda^2} A_\lambda^{(1)}(\hat{\mathcal{F}}_{1,\lambda}, \hat{\mathcal{F}}_{2,\lambda}) \right] \\
& = \frac{\hat{b}}{2\lambda^2} \int \frac{\Lambda Q_\lambda + 2Q_\lambda}{Q_\lambda^2} \left[\frac{\hat{w}}{\lambda^2} \mathcal{L}_\lambda^{(1)}(\hat{\mathcal{F}}_{1,\lambda}, \hat{\mathcal{F}}_{2,\lambda}) + \frac{\hat{w}_2}{\lambda^2} \hat{\mathcal{F}}_{1,\lambda} \right] \\
& - \frac{\hat{b}}{\lambda^2} \left\{ \int \hat{w}_1 \cdot \nabla \hat{\mathcal{F}}_{2,\lambda} + \int Q \nabla \mathcal{M}_\lambda^{(1)}(\hat{\mathcal{F}}_{1,\lambda}, \hat{\mathcal{F}}_{2,\lambda}) \cdot \nabla \hat{z} \right\}. \tag{5.27}
\end{aligned}$$

Injecting (5.26) and (5.27) in (5.25) yields:

$$\begin{aligned}
& \frac{d}{dt} \left\{ \int \frac{\hat{b}(\Lambda Q + 2Q)_\lambda}{2\lambda^2 Q_\lambda^2} \hat{w}_2 \hat{w} - \int \frac{\hat{b} \hat{w}_1 \nabla \hat{z}}{\lambda^2} \right\} = \int \frac{\hat{b}(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \hat{w}_2^2 + \frac{\hat{b}}{\lambda^2} \int \hat{w}_2 \mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2) \\
& + \int \frac{\hat{w}_1 \nabla \hat{z}}{\lambda^2} \left[4\hat{b} \frac{\lambda_s}{\lambda} - \hat{b}_s \right] - \frac{\hat{b}}{2\lambda^2} \int Q_\lambda \nabla (\mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2)) \cdot \nabla \left[2\hat{z} + \frac{(2Q + \Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w} \right] \\
& + \frac{\hat{b}}{2\lambda^2} \int \frac{\Lambda Q_\lambda + 2Q_\lambda}{Q_\lambda^2} \left[\frac{\hat{w}}{\lambda^2} \mathcal{L}_\lambda^{(1)}(\hat{\mathcal{F}}_{1,\lambda}, \hat{\mathcal{F}}_{2,\lambda}) + \frac{\hat{w}_2}{\lambda^2} \hat{\mathcal{F}}_{1,\lambda} \right] \\
& - \frac{\hat{b}}{\lambda^2} \left\{ \int \hat{w}_1 \cdot \nabla \hat{\mathcal{F}}_{2,\lambda} + \int Q \nabla \mathcal{M}_\lambda^{(1)}(\hat{\mathcal{F}}_{1,\lambda}, \hat{\mathcal{F}}_{2,\lambda}) \cdot \nabla \hat{z} \right\} \\
& + \int \frac{\hat{b}(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \hat{w} \nabla \cdot V_\lambda^{(1)}(\hat{w}, \hat{y}) - 2\hat{b} \int \frac{\hat{w}_1}{\lambda^2 Q_\lambda} V_\lambda^{(1)}(\hat{w}, \hat{z}) \\
& + O\left(\int \frac{|\hat{w}_2 \hat{w}| + |\hat{w}_1|^2}{\lambda^2 Q(1+r^2)} \left[|\hat{b}_s| + \hat{b} \left| \frac{\lambda_s}{\lambda} \right| \right] \right). \tag{5.28}
\end{aligned}$$

To conclude, summing the above equality with (5.17) yields the modified energy identity:

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left\{ \langle \mathcal{M}_\lambda \widehat{\mathbf{W}}_2, \widehat{\mathbf{W}}_2 \rangle + \int \frac{\hat{b}(\Lambda Q + 2Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w}_2 \hat{w} - 2 \int \frac{\hat{b} \hat{w}_1 \nabla \hat{z}}{\lambda^2} \right\} \\
& = - \int Q_\lambda |\nabla \mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2)|^2 - \int |\Delta \mathcal{M}_\lambda^{(2)}(\hat{w}_2, \hat{z}_2)|^2 + \frac{\hat{b}}{\lambda^2} \int \hat{w}_2 \mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2) \\
& + \int \frac{\hat{w}_1 \nabla \hat{z}}{\lambda^2} \left[4\hat{b} \frac{\lambda_s}{\lambda} - \hat{b}_s \right] - \frac{\hat{b}}{2\lambda^2} \int Q_\lambda \nabla (\mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2)) \cdot \nabla \left[2\hat{z} + \frac{(2Q + \Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w} \right] \\
& + \frac{\hat{b}}{2\lambda^2} \int \frac{\Lambda Q_\lambda + 2Q_\lambda}{Q_\lambda^2} \left[\frac{\hat{w}}{\lambda^2} \mathcal{L}_\lambda^{(1)}(\hat{\mathcal{F}}_{1,\lambda}, \hat{\mathcal{F}}_{2,\lambda}) + \frac{\hat{w}_2}{\lambda^2} \hat{\mathcal{F}}_{1,\lambda} \right] + \left\langle \widehat{\mathbf{W}}_2, \frac{1}{\lambda^2} \mathcal{M}_\lambda \mathcal{L}_\lambda \widehat{\mathcal{F}}_\lambda \right\rangle \\
& - \frac{\hat{b}}{\lambda^2} \left\{ \int \hat{w}_1 \cdot \nabla \hat{\mathcal{F}}_{2,\lambda} + \int Q \nabla \mathcal{M}_\lambda^{(1)}(\hat{\mathcal{F}}_{1,\lambda}, \hat{\mathcal{F}}_{2,\lambda}) \cdot \nabla \hat{z} \right\} \\
& + \int \frac{\hat{b}(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \hat{w} \nabla \cdot V_\lambda^{(1)}(\hat{w}, \hat{y}) - 2\hat{b} \int \frac{\hat{w}_1}{\lambda^2 Q_\lambda} V_\lambda^{(1)}(\hat{w}, \hat{z}) - \langle V_\lambda \widehat{\mathbf{W}}, \nabla \mathcal{M}_\lambda \widehat{\mathbf{W}}_2 \rangle \\
& + \int \frac{\hat{w}_2^2}{2\lambda^2 Q_\lambda^2} \left[\left(\hat{b} + \frac{\lambda_s}{\lambda} \right) (\Lambda Q)_\lambda \right] + O\left(\int \frac{|\hat{w}_2 \hat{w}| + |\hat{w}_1|^2}{\lambda^2 Q(1+r^2)} \left[|\hat{b}_s| + \hat{b} \left| \frac{\lambda_s}{\lambda} \right| \right] \right). \tag{5.29}
\end{aligned}$$

Now, all terms are manageable. We treat each term in the rest of this section. We shall make an intensive use of the bounds of the Lemma 4.8 and the interpolation bound of the Proposition C.1.

5.3. Boundary terms in time. We must verify that this both terms aren't bigger than the quantity which we would like to control.

$$\begin{aligned}
\left| \int \frac{\hat{b}(\Lambda Q + 2Q)}{\lambda^2 Q^2} \hat{w}_2 \hat{w} \right| & \lesssim \frac{b}{\lambda^4} \int \frac{|\hat{\varepsilon}_2 \hat{\varepsilon}|}{(1+r^2)Q} \lesssim \frac{b}{\lambda^4} \left(\int \frac{\varepsilon_2^2}{Q} \right)^{\frac{1}{2}} \left(\int \varepsilon^2 \right)^{\frac{1}{2}} \\
& \lesssim \frac{C(M)b}{\lambda^4} \left[\|\varepsilon_2\|_{L^2_Q}^2 + \|\zeta_2\|_{L^2_Q}^2 + \|\zeta\|_{L^2}^2 + \|\varepsilon\|_{L^2}^2 \right] \\
& \lesssim \frac{C(M)b^3}{\lambda^4 |\log b|^2}
\end{aligned}$$

The control of the second term is more delicate.

$$\left| \frac{1}{\lambda^2} \int b \hat{w}_1 \cdot \nabla \hat{z} \right| \lesssim \frac{b}{\lambda^4} \left(\int \nabla \hat{\varepsilon} \nabla \hat{\eta} - \int \frac{\nabla Q}{Q} \hat{\varepsilon} \nabla \hat{\eta} + \int Q |\nabla \hat{\eta}|^2 \right) \quad (5.30)$$

Now,

$$\begin{aligned} \int Q |\nabla \hat{\eta}|^2 &\lesssim \left(\int |\nabla \eta|^2 \int \frac{|\nabla \eta|^2}{1 + \tau^8} \right)^{\frac{1}{2}} + \int \frac{\nabla \xi^2}{1 + \tau^2} \\ &\lesssim \|\eta\|_{\dot{H}^1} \|\mathbf{E}_2\|_{X_Q} + \int \frac{\nabla \xi^2}{1 + \tau^2} \end{aligned} \quad (5.31)$$

$$\lesssim K^* b^{\frac{5}{2}} |\log b|^2 + C(M) \frac{b^2}{|\log b|^2} \lesssim C(M) \frac{b^2}{|\log b|^2} \quad (5.32)$$

Hence

$$\begin{aligned} \left| \int \nabla \hat{\varepsilon} \nabla \hat{\eta} - \int \frac{\nabla Q}{Q} \hat{\varepsilon} \nabla \hat{\eta} \right| &\lesssim \left(\left\{ \int (1 + \tau^2) |\nabla \hat{\varepsilon}|^2 + \int |\hat{\varepsilon}|^2 \right\} \int Q |\nabla \hat{\eta}|^2 \right)^{\frac{1}{2}} \\ &\lesssim C(M) \frac{b}{|\log b|} \left[\int (1 + \tau^2) |\nabla \varepsilon|^2 + \int |\varepsilon|^2 + \int (1 + \tau^2) |\nabla \zeta|^2 + \int |\zeta|^2 \right]^{\frac{1}{2}} \\ &\lesssim C(M) \frac{b}{|\log b|} \left[\|\mathbf{E}_2\|_{X_Q}^2 + \frac{b^2}{|\log b|^2} \right]^{\frac{1}{2}} \lesssim C(M) \frac{b^2}{|\log b|^2} \end{aligned} \quad (5.33)$$

Injecting (5.31), (5.33) in (5.30) yields

$$\left| \frac{1}{\lambda^2} \int b \hat{w}_1 \cdot \nabla \hat{z} \right| \lesssim C(M) \frac{b^3}{\lambda^4 |\log b|^2} \quad (5.34)$$

□

5.4. Quadratic terms. We aren't in position to treat the term $\frac{\hat{b}}{\lambda^2} \int \hat{w}_2 \mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2)$, which has a wrong sign. Indeed, as $\int \hat{w}_2 = 0$, \mathcal{M} is a positive operator. We shall treat this term in the conclusion. However, both terms $-\int Q_\lambda |\nabla \mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2)|^2 - \int |\mathcal{M}_\lambda^{(2)}(\hat{w}_2, \hat{z}_2)|^2$ are non positive, and they shall be very helpful to estimate some terms in the rest of this proof.

Now, with the modulation equations, we have proved that :

$$\left| \frac{\lambda_s}{\lambda} \right| \lesssim b, \quad |\hat{b}_s| \lesssim b^2.$$

Thus,

$$\left| \int \frac{\hat{\varepsilon}_1 \cdot \nabla \hat{z}}{2\lambda^6} \left[8\hat{b} \frac{\lambda_s}{\lambda} - 2\hat{b}_s \right] \right| \lesssim \frac{b^2}{\lambda^6} \int |\hat{\varepsilon}_1 \cdot \nabla \hat{\eta}| \lesssim \frac{C(M)b^4}{\lambda^6 |\log b|^2}.$$

We have still proved the last inequality in the last subsection. In the same way, we can control the error term:

$$\begin{aligned} &\int \frac{|\hat{\varepsilon}_2 \hat{\varepsilon}| + |\hat{\varepsilon}_1|^2}{\lambda^6 Q (1 + \tau^2)} (|b_s| + b) \frac{\lambda_s}{\lambda} + b + b^2 \\ &\lesssim \frac{b^2}{\lambda^6} \int [(1 + \tau^4) \hat{\varepsilon}_2^2 + (1 + \tau^2) |\hat{\varepsilon}_1|^2 + \hat{\varepsilon}^2] \\ &\lesssim C(M) \frac{b^2}{\lambda^6} \left[\|\mathbf{E}_2\|_{X_Q}^2 + \frac{b^2}{|\log b|^2} \right]. \end{aligned}$$

Next, we can estimate this term :

$$\begin{aligned}
& \left| \frac{\hat{b}}{2\lambda^2} \int Q_\lambda \nabla(\mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2)) \cdot \nabla \left[2\hat{z} + \frac{(2Q + \Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w} \right] \right| \\
& \leq \frac{1}{100} \int Q_\lambda |\nabla(\mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2))|^2 + \frac{Cb^2}{\lambda^6} \int Q \left[|\nabla \hat{\eta}|^2 + \frac{|\nabla \hat{\varepsilon}|^2}{(1 + \tau^4)Q^2} + \frac{\hat{\varepsilon}^2}{(1 + \tau^6)Q^2} \right] \\
& \leq \frac{1}{100} \int Q_\lambda |\nabla(\mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2))|^2 + C(M) \frac{b^2}{\lambda^6} \left[\|\mathbf{E}_2\|_{X_Q}^2 + \frac{b^2}{|\log b|^2} \right].
\end{aligned}$$

We are focusing now on all terms depending on the potential V_λ . We recall that:

$$V_\lambda \widehat{\mathbf{W}} = [\partial_t, A_\lambda] \widehat{\mathbf{W}} = \frac{\lambda_s}{\lambda} \begin{pmatrix} (\nabla \phi_{\Lambda Q})_\lambda \hat{w} + (\Lambda Q)_\lambda \nabla \hat{z} \\ 0 \end{pmatrix}.$$

and thus, we have the following bound

$$|V_\lambda \widehat{\mathbf{W}}| \lesssim \frac{b}{\lambda^2} \left[\frac{\hat{\varepsilon}}{1 + \tau^3} + \frac{\nabla \hat{\eta}}{1 + \tau^4} \right] \quad (5.35)$$

Hence,

$$\begin{aligned}
& \left| \int \frac{\hat{b}(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \hat{w} \nabla \cdot V_\lambda^{(1)}(\hat{w}, \hat{y}) \right| \lesssim \frac{b^2}{\lambda^6} \int (1 + \tau^4) \hat{\varepsilon} \left[\frac{\hat{\varepsilon}}{1 + \tau^4} + \frac{\nabla \hat{\eta}}{1 + \tau^5} + \frac{\nabla \hat{\varepsilon}}{1 + \tau^3} + \frac{\nabla^2 \hat{\eta}}{1 + \tau^4} \right] \\
& \lesssim \frac{b^2}{\lambda^6} \left[\|\mathbf{E}_2\|_{X_Q}^2 + \int |\zeta|^2 + \int (1 + \tau^2) |\nabla \zeta|^2 + \int \frac{|\nabla \xi|^2}{1 + \tau^2} + \int |\Delta \xi|^2 \right] \\
& \lesssim \frac{b^2}{\lambda^6} \left[\|\mathbf{E}_2\|_{X_Q}^2 + \frac{b^2}{|\log b|^2} \right]
\end{aligned}$$

Next,

$$\begin{aligned}
\left| \hat{b} \int \frac{\hat{w}_1}{\lambda^2 Q_\lambda} V_\lambda^{(1)}(\hat{w}, \hat{z}) \right| & \lesssim \frac{b^2}{\lambda^6} \int (1 + \tau^4) \nabla \hat{\varepsilon} \left[\frac{\hat{\varepsilon}}{1 + \tau^3} + \frac{\nabla \hat{\eta}}{1 + \tau^4} \right] \\
& \lesssim \frac{b^2}{\lambda^6} \left[\|\mathbf{E}_2\|_{X_Q}^2 + \int |\zeta|^2 + \int (1 + \tau^2) |\nabla \zeta|^2 + \int \frac{|\nabla \xi|^2}{1 + \tau^2} \right] \\
& \lesssim \frac{b^2}{\lambda^6} \left[\|\mathbf{E}_2\|_{X_Q}^2 + \frac{b^2}{|\log b|^2} \right]
\end{aligned}$$

As $V_\lambda^{(2)}(\hat{w}_2, \hat{z}_2) = 0$, we have for the last term depending on V_λ

$$\begin{aligned}
& \left| \left\langle V_\lambda \widehat{\mathbf{W}}, \nabla \mathcal{M}_\lambda \widehat{\mathbf{W}}_2 \right\rangle \right| = \left| \int V_l^{(2)}(\hat{w}_2, \hat{z}_2) \nabla \mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2) \right| \\
& \leq \frac{1}{100} \int Q |\nabla \mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2)|^2 + O \left(\frac{b}{\lambda^2} \int (1 + \tau^4) \left[\frac{|\hat{\varepsilon}|^2}{1 + \tau^6} + \frac{|\nabla \hat{\eta}|^2}{1 + \tau^6} \right] \right) \\
& \leq \frac{1}{100} \int Q |\nabla \mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2)|^2 + \frac{b}{\lambda^2} O \left(\|\mathbf{E}_2\|_{X_Q}^2 + \int |\zeta|^2 + \int \frac{|\nabla \xi|^2}{1 + \tau^2} \right) \\
& \leq \frac{1}{100} \int Q |\nabla \mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2)|^2 + C(M) \frac{b^2}{\lambda^6} \left[\|\mathbf{E}_2\|_{X_Q}^2 + \frac{b^2}{|\log b|^2} \right]
\end{aligned}$$

To conclude this subsection dedicated to the quadratic terms, we must estimate the last term. We must carefully study this term because of the very bad estimation coming from (4.36) and (4.47):

$$\left| \hat{b} + \frac{\lambda_s}{\lambda} \right| \lesssim \frac{b}{|\log b|}.$$

We recall the decomposition of the gap Ξ using in the proof of the lemma (4.8):

$$\Xi_2 = \mathcal{L}\Xi = \mathcal{L}\Xi_{sm} + (b - \hat{b})\Lambda Q + \mathcal{R}$$

with the estimation

$$\mathcal{R} = (b - \hat{b}) \begin{vmatrix} O\left(\frac{1_{r \geq B_1}}{1+r^4}\right) \\ O\left(\frac{1_{r \geq B_1}}{1+r^2}\right) \end{vmatrix}$$

Hence,

$$\begin{aligned} & \left| \int \frac{\hat{w}_2^2}{\lambda^2 Q_\lambda^2} \left[\left(\frac{\lambda_s}{\lambda} + \hat{b} \right) (\Lambda Q)_\lambda \right] \right| \lesssim \frac{b}{|\log b| \lambda^6} \left| \int \frac{(\varepsilon_2 + \mathcal{L}^{(1)}\Xi_{sm} + \mathcal{R}^{(1)} + (b - \hat{b})\Lambda Q)^2}{Q^2} \Lambda Q \right| \\ & \lesssim \frac{b}{|\log b| \lambda^6} \left[\|\mathbf{E}_2\|_{X_Q}^2 + \|\mathcal{L}\Xi_{sm}\|_{X_Q}^2 + \|\mathcal{R}\|_{X_Q}^2 + \frac{b^2}{|\log b|^2} \left| \int \left(\frac{\Lambda Q}{Q} \right)^2 \Lambda Q \right| \right] \\ & + \frac{b}{|\log b|} \left| \int (\varepsilon_2 + \mathcal{L}^{(1)}\Xi_{sm}) \frac{(\Lambda Q)^2}{Q^2} \right| + \frac{b^2}{|\log b|^2} \int_{r \geq B_1} \frac{1}{1+r^4} \\ & \lesssim \frac{b}{\lambda^6} \left[\frac{b^{\frac{3}{2}}}{|\log b|} \|\mathbf{E}_2\|_{X_Q} + \frac{b^3}{|\log b|^2} + \frac{b^2}{|\log b|^2} \left| \int \left(\frac{\Lambda Q}{Q} \right)^2 \Lambda Q \right| + \frac{b}{|\log b|^2} \left| \int (\varepsilon_2 + \mathcal{L}^{(1)}\Xi_{sm}) \frac{(\Lambda Q)^2}{Q^2} \right| \right]. \end{aligned}$$

Now, using that $\Lambda^2 Q = \nabla \cdot (r \Lambda Q)$, we have that

$$\nabla \phi_{\Lambda^2 Q} = r \Lambda Q$$

Now, using the radial representation of Poisson field, we have:

$$\phi_{\Lambda^2 Q}(0) = \int_0^\infty \Lambda^2 Q \log r r dr = 0.$$

Thus,

$$\phi_{\Lambda^2 Q}(r) = \int_0^r r \nabla \cdot (r Q) r dr = r^2 Q.$$

Using the explicit formula of $\Lambda^i Q$, for $0 \leq i \leq 2$, we obtained:

$$\mathcal{M}(\Lambda^2 Q, r^2 Q) = \begin{vmatrix} \frac{\Lambda^2 Q}{r^2 Q} + r^2 Q & \\ r^2 Q - \phi_{\Lambda^2 Q} & 0 \end{vmatrix} = \begin{vmatrix} \left(\frac{\Lambda Q}{Q} \right)^2 & \\ 0 & \end{vmatrix} \quad (5.36)$$

We are now in position to compute:

$$\int \left(\frac{\Lambda Q}{Q} \right)^2 \Lambda Q = \langle \mathcal{M}(\Lambda^2 Q, r^2 Q), \Lambda Q \rangle = \left\langle \begin{vmatrix} \Lambda^2 Q \\ r^2 Q \end{vmatrix}, \mathcal{M}(\Lambda Q) \right\rangle = -2 \int \Lambda^2 Q = 0.$$

We recall the following degeneracy :

$$\left(\frac{\Lambda Q}{Q} \right)^2 = 4 + O\left(\frac{1}{1+\tau^2} \right)$$

Hence, with (4.58)

$$\left| \int \mathcal{L}^{(1)}\Xi_{sm} \frac{(\Lambda Q)^2}{Q^2} \right| \lesssim \int \frac{|\mathcal{L}^{(1)}\Xi_{sm}|}{1+\tau^2} \lesssim \|\mathcal{L}^{(1)}\Xi_{sm}\|_{L^2} \lesssim \frac{b^2}{|\log b|}.$$

Finally, using the cancellation $\int \varepsilon_2 = 0$, we obtain

$$\begin{aligned} \int \varepsilon_2 \frac{(\Lambda Q)^2}{Q^2} &= \langle \mathbf{E}_2, \mathcal{M}(\Lambda^2 \mathbf{Q} + 2\Lambda \mathbf{Q}) \rangle = \langle \mathcal{M} \mathbf{E}_2, \Lambda^2 \mathbf{Q} + 2\Lambda \mathbf{Q} \rangle \\ &= \int \mathcal{M}^{(1)}(\varepsilon_2, \eta_2) \nabla \cdot (r(\Lambda Q + 2Q)) + \int \nabla \mathcal{M}^{(2)}(\varepsilon_2, \eta_2) \nabla \left(r^2 Q + \frac{8}{1+r^2} \right) \\ &= - \int \nabla \mathcal{M}^{(1)}(\varepsilon_2, \eta_2) r(\Lambda Q + 2Q) - \int \Delta \mathcal{M}^{(2)}(\varepsilon_2, \eta_2) \left(r^2 Q + \frac{8}{1+r^2} \right) \end{aligned}$$

Thus,

$$\begin{aligned} \left| \int \varepsilon_2 \frac{(\Lambda Q)^2}{Q^2} \right| &\lesssim \left(\int Q |\nabla(\mathcal{M}^{(1)}(\varepsilon_2, \eta_2))|^2 \right)^{\frac{1}{2}} \left(\int \frac{r^2 (\Lambda Q + 2Q)^2}{Q} \right)^{\frac{1}{2}} \\ &\quad + \left(\int |\Delta(\mathcal{M}^{(2)}(\varepsilon_2, \eta_2))|^2 \right)^{\frac{1}{2}} \left(\int \left[r^2 Q + \frac{8}{1+r^2} \right]^2 \right)^{\frac{1}{2}} \\ &\lesssim \left[\int Q |\nabla(\mathcal{M}^{(1)}(\hat{\varepsilon}_2, \hat{\eta}_2))|^2 + \int |\Delta(\mathcal{M}^{(2)}(\hat{\varepsilon}_2, \hat{\eta}_2))|^2 + \frac{b^4}{|\log b|^2} \right]^{\frac{1}{2}}, \end{aligned}$$

where we have use (4.65) in the last line. Hence,

$$\begin{aligned} \frac{b^2}{\lambda^6 |\log b|^2} \left| \int \varepsilon_2 \frac{(\Lambda Q)^2}{Q^2} \right| &\leq \frac{1}{100\lambda^6} \left| \int \varepsilon_2 \frac{(\Lambda Q)^2}{Q^2} \right|^2 + O\left(\frac{b^4}{\lambda^6 |\log b|^4} \right) \\ &\leq \frac{1}{100\lambda^6} \left[\int Q |\nabla(\mathcal{M}^{(1)}(\hat{\varepsilon}_2, \hat{\eta}_2))|^2 + \int |\Delta(\mathcal{M}^{(2)}(\hat{\varepsilon}_2, \hat{\eta}_2))|^2 \right] + O\left(\frac{b^4}{\lambda^6 |\log b|^4} \right) \end{aligned}$$

The collection above bounds yields the admissible control:

$$\begin{aligned} &\left| \int \frac{\hat{w}_2^2}{\lambda^2 Q_\lambda^2} \left[\left(\frac{\lambda_s}{\lambda} + \hat{b} \right) (\Lambda Q)_\lambda \right] \right| \\ &\lesssim \frac{b}{\lambda^6} \left[b^{\frac{3}{2}} \|\mathbf{E}_2\|_{X_Q} + \frac{b^3}{|\log b|^2} \right] + \frac{1}{100\lambda^6} \left[\int Q |\nabla(\mathcal{M}^{(1)}(\hat{\varepsilon}_2, \hat{\eta}_2))|^2 + \int |\Delta(\mathcal{M}^{(2)}(\hat{\varepsilon}_2, \hat{\eta}_2))|^2 \right]. \end{aligned}$$

To conclude the proof of the Proposition 5.1, we shall focus on the term depending on $\widehat{\mathcal{F}}_\lambda$. We recall that:

$$\widehat{\mathcal{F}}_\lambda = \left(\widehat{\mathbf{Mod}} \right)_\lambda + \mathbf{H}_\lambda + \mathbf{G}_\lambda + \left(\tilde{\Psi}_b \right)_\lambda + \left(\Theta_b(\varepsilon, \eta) \right)_\lambda + \left(\mathbf{N}(\varepsilon, \eta) \right)_\lambda \quad (5.37)$$

As the dependence on $\widehat{\mathcal{F}}_\lambda$ in (5.29) is linear, we shall estimate each term of the decomposition (5.37) in separate subsections.

5.5. $\tilde{\Psi}_b$ terms. First, we use the bounds (3.112) and (3.113) coming from the Proposition 3.2 to estimate:

$$\begin{aligned} &\left| \left\langle \widehat{\mathbf{W}}_2, \mathcal{M}_\lambda \mathcal{L}_\lambda \tilde{\Psi}_b \right\rangle \right| = \left| \left\langle \mathcal{M}_\lambda \widehat{\mathbf{W}}_2, \mathcal{L}_\lambda \tilde{\Psi}_b \right\rangle \right| \\ &\lesssim \left| \int \nabla \mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2) Q \nabla \mathcal{M}_\lambda^{(1)}(\tilde{\Psi}_{b,\lambda}^{(1)}, \tilde{\Psi}_{b,\lambda}^{(2)}) \right| + \left| \int \nabla \mathcal{M}_\lambda^{(2)}(\hat{w}_2, \hat{z}_2) \nabla \mathcal{L}_\lambda^{(2)}(\tilde{\Psi}_{b,\lambda}^{(1)}, \tilde{\Psi}_{b,\lambda}^{(2)}) \right| \\ &\leq \frac{1}{100} \left[\int Q |\nabla(\mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2))|^2 + \int |\Delta(\mathcal{M}_\lambda^{(2)}(\hat{w}_2, \hat{z}_2))|^2 \right] \\ &\quad + O\left(\frac{1}{\lambda^6} \left[\int Q |\nabla \mathcal{M}_\lambda^{(1)}(\tilde{\Psi}_{b,\lambda}^{(1)}, \tilde{\Psi}_{b,\lambda}^{(2)})|^2 + \int |\mathcal{L}_\lambda^{(2)}(\tilde{\Psi}_{b,\lambda}^{(1)}, \tilde{\Psi}_{b,\lambda}^{(2)})|^2 \right] \right) \\ &\leq \frac{1}{100} \left[\int Q |\nabla(\mathcal{M}_\lambda^{(1)}(\hat{w}_2, \hat{z}_2))|^2 + \int |\Delta(\mathcal{M}_\lambda^{(2)}(\hat{w}_2, \hat{z}_2))|^2 \right] + O\left(\frac{b^4}{\lambda^6 |\log b|^2} \right). \end{aligned}$$

Now, with the bound (3.111) and the degeneracy (5.21), we obtain

$$\begin{aligned}
& \left| \frac{\hat{b}}{2\lambda^2} \int \frac{\Lambda Q_\lambda + 2Q_\lambda}{Q_\lambda^2} \left[\frac{\hat{w}}{\lambda^2} \mathcal{L}^{(1)} \left(\tilde{\Psi}_{b,\lambda}^{(1)}, \tilde{\Psi}_{b,\lambda}^{(2)} \right) \right] \right| \\
& \lesssim \frac{b}{\lambda^6} \int (1 + \tau^2) \left\{ \hat{\varepsilon} \mathcal{L}^{(1)} \left(\tilde{\Psi}_b^{(1)}, \tilde{\Psi}_b^{(2)} \right) + \hat{\varepsilon}_2 \tilde{\Psi}_b^{(1)} \right\} \\
& \lesssim \frac{b}{\lambda^6} \left[\left\| \mathcal{L}^{(1)} \left(\tilde{\Psi}_b^{(1)}, \tilde{\Psi}_b^{(2)} \right) \right\|_{L_Q^2} \|\hat{\varepsilon}\|_{L^2} + \|\tilde{\Psi}_b^{(1)}\|_{L^2} \|\hat{\varepsilon}_2\|_{L_Q^2} \right] \\
& \lesssim \frac{b}{\lambda^6} \frac{b^2}{|\log b|} \left[\|\mathbf{E}_2\|_{X_Q} + \|\zeta\|_{L^2} + \|\zeta\|_{L_Q^2} \right] \lesssim \frac{b}{\lambda^6} \frac{b^2}{|\log b|} \left[\|\mathbf{E}_2\|_{X_Q} + \frac{b}{|\log b|} \right]
\end{aligned}$$

With (3.112) we can control the following term

$$\begin{aligned}
& \left| \frac{b}{\lambda^6} \int \hat{\varepsilon}_1 \nabla \tilde{\Psi}_b^{(2)} \right| \lesssim \frac{b}{\lambda^6} \left(\int (1 + \tau^2) |\hat{\varepsilon}_1|^2 \int \frac{|\nabla \tilde{\Psi}_b^{(2)}|^2}{1 + \tau^2} \right)^{\frac{1}{2}} \\
& \lesssim \frac{b}{\lambda^6} \frac{b^{\frac{5}{2}}}{|\log b|} \left[C(M) \|\mathbf{E}_2\|_{X_Q} + \left(\int (1 + \tau^2) |\nabla \zeta|^2 + \int |\zeta|^2 + \int \frac{|\nabla \xi|^2}{1 + r^2} \right)^{\frac{1}{2}} \right] \\
& \lesssim \frac{b}{\lambda^6} \left[b^2 \|\mathbf{E}_2\|_{X_Q} + \frac{b^3}{|\log b|^2} \right].
\end{aligned}$$

We recall the bound (5.31)

$$\int Q |\nabla \hat{\eta}|^2 \lesssim C(M) \frac{b^2}{|\log b|}$$

Using this bound together the bound (3.113) :

$$\begin{aligned}
& \left| \frac{b}{\lambda^6} \int \nabla \hat{\eta} Q \nabla \mathcal{M}^{(1)} \left(\tilde{\Psi}_b^{(1)}, \tilde{\Psi}_b^{(2)} \right) \right| \lesssim \frac{b}{\lambda^6} \left(\int Q |\nabla \hat{\eta}|^2 \int Q \left| \nabla \mathcal{M}^{(1)} \left(\tilde{\Psi}_b^{(1)}, \tilde{\Psi}_b^{(2)} \right) \right|^2 \right)^{\frac{1}{2}} \\
& \lesssim C(M) \frac{b}{\lambda^6} \frac{b^4}{|\log b|^2}
\end{aligned}$$

5.6. \mathbf{G} terms. We recall that $\mathbf{G} = -c_b b^2 \check{\mathbf{T}}$ with

$$\left| \begin{array}{c} \check{T}^{(1)} \\ \nabla \check{T}^{(2)} \end{array} \right| = \chi_{\frac{B_0}{4}} \left| \begin{array}{c} T_1 \\ \nabla S_1 \end{array} \right|.$$

First, using that $\mathcal{L}\mathbf{T}_1 = \Lambda\mathbf{Q}$, the cancellation $\int \hat{\varepsilon}_2 = 0$ and $\mathcal{M}\Lambda\mathbf{Q} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$, we obtain :

$$\langle \hat{\mathbf{E}}_2, \mathcal{M}\mathcal{L}\check{\mathbf{T}} \rangle = - \langle \hat{\mathbf{E}}_2, \mathcal{M}\mathcal{L}[\mathbf{T}_1 - \check{\mathbf{T}}] \rangle$$

Moreover,

$$\hat{\mathbf{E}}_2 = \mathbf{E}_2 + \mathcal{L}\Xi_{sm} + (b - \hat{b})\Lambda\mathbf{Q} + \mathcal{R}.$$

where \mathcal{R} satisfies the bound (4.69). With this decomposition, we obtain:

$$\langle \hat{\mathbf{E}}_2, \mathcal{M}\mathcal{L}\check{\mathbf{T}} \rangle = - \langle \mathbf{E}_2 + \mathcal{L}\Xi_{sm} + \mathcal{R}, \mathcal{M}\mathcal{L}[\mathbf{T}_1 - \check{\mathbf{T}}] \rangle$$

We compute now $\mathcal{L}[\mathbf{T}_1 - \check{\mathbf{T}}]$:

$$\mathcal{L}[\mathbf{T}_1 - \check{\mathbf{T}}] = \begin{vmatrix} \nabla \cdot \{ \nabla(T_1(1 - \chi_{\frac{B_0}{4}})) + T_1 \nabla S_1(1 - \chi_{\frac{B_0}{4}})^2 \} \\ \nabla \cdot \{ (\nabla S_1)(1 - \chi_{\frac{B_0}{4}}) \} - (1 - \chi_{\frac{B_0}{4}}) T_1 \end{vmatrix}$$

Hence :

$$|\mathcal{L}^{(1)}[\mathbf{T}_1 - \check{\mathbf{T}}]| \lesssim \frac{1}{1 + \tau^4} \mathbf{1}_{\tau \geq \frac{B_0}{4}}.$$

and

$$\begin{aligned} \nabla \mathcal{M}^{(2)}(\mathcal{L}[\mathbf{T}_1 - \check{\mathbf{T}}]) &= -(\chi_{\frac{B_0}{4}})' \phi_{\Lambda Q} + ((\chi_{\frac{B_0}{4}})' \nabla S_1)' - \chi_{\frac{B_0}{4}}' T_1 - T_1 \nabla S_1 \chi_{\frac{B_0}{4}} (1 - \chi_{\frac{B_0}{4}}) \\ &\quad - (1 - \chi_{\frac{B_0}{4}}) \{ \nabla \phi_{\Lambda Q} - \nabla T_1 - T_1 \nabla S_1 \} \end{aligned}$$

By construction of the profiles T_1 and S_1 , we have

$$\nabla \phi_{\Lambda Q} - \nabla T_1 - T_1 \nabla S_1 = 0.$$

Hence, we obtain the estimation

$$|\nabla \mathcal{M}^{(2)}(\mathcal{L}[\mathbf{T}_1 - \check{\mathbf{T}}])| \lesssim \frac{\sqrt{b}}{r} \mathbf{1}_{\frac{B_0}{4} \leq \tau \leq \frac{B_0}{2}}.$$

Thus

$$\begin{aligned} \langle \widehat{\mathbf{W}}_2, \mathcal{M}_\lambda \mathcal{L}_\lambda \mathbf{G}_\lambda \rangle &\lesssim \frac{b^2}{\lambda^6 |\log b|} \left| \langle \mathbf{E}_2 + \mathcal{L} \Xi_{sm} + \mathcal{R}, \mathcal{M} \mathcal{L}[\mathbf{T}_1 - \check{\mathbf{T}}] \rangle \right| \\ &\lesssim \frac{b^2}{\lambda^6 |\log b|} \left(\|\varepsilon_2\|_{L^2_Q} + \|\mathcal{L}^{(1)} \Xi_{sm}\|_{L^2_Q} + \|\mathcal{R}^{(1)}\|_{L^2_Q} \right) \|\mathcal{L}^{(1)}[\mathbf{T}_1 - \check{\mathbf{T}}]\|_{L^2_Q} \\ &\quad + \frac{b^2}{\lambda^6 |\log b|} \left(\|\nabla \eta_2\|_{L^2} + \|\mathcal{L}^{(2)} \Xi_{sm}\|_{L^2} + \|\mathcal{R}^{(2)}\|_{L^2} \right) \|\nabla \mathcal{M}^{(2)}(\mathcal{L}[\mathbf{T}_1 - \check{\mathbf{T}}])\|_{L^2} \\ &\lesssim \frac{b^{\frac{5}{2}}}{\lambda^6 |\log b|} (\|\mathbf{E}_2\|_{X_Q} + \|\mathcal{L} \Xi_{sm}\|_{X_Q} + \|\mathcal{R}\|_{X_Q}) \lesssim \frac{b^{\frac{5}{2}}}{\lambda^6 |\log b|} \left(\|\mathbf{E}_2\|_{X_Q} + \frac{b^{\frac{3}{2}}}{|\log b|} \right). \end{aligned}$$

Other terms depending on \mathbf{G} in (5.29) can be treated in brute force. Indeed, using (C.1) and Lemma 4.8:

$$\begin{aligned} b \left| \int \frac{\Lambda Q + 2Q}{Q^2} (\hat{\varepsilon} \mathcal{L}^{(1)} \mathbf{G} + \hat{\varepsilon}_2 G^{(1)}) \right| &\lesssim b \frac{b^2}{|\log b|} \int \frac{1}{(1+r^2)Q} \left[\frac{|\hat{\varepsilon}|}{1+r^4} + \frac{|\hat{\varepsilon}_2|}{1+r^2} \right] \\ &\lesssim b \frac{b^2}{|\log b|} \left[\|\varepsilon\|_{L^2} + \|\varepsilon_2\|_{L^2_Q} + \|\zeta_2\|_{L^2_Q} + \|\zeta\|_{L^2} \right] \\ &\lesssim b \left[C(M) \frac{b^2}{|\log b|} \|\mathbf{E}_2\|_{X_Q} + \frac{b^3}{|\log b|^2} \right]. \end{aligned}$$

Similarly, using (C.2) and Lemma 4.8:

$$\begin{aligned} b \left| \int \hat{\varepsilon}_1 \cdot \nabla G^{(2)} \right| &\lesssim b \frac{b^2}{|\log b|} \int \frac{1 + |\log r|}{1+r} \left[|\varepsilon_1| + |\nabla \zeta| + Q |\nabla \xi| + \frac{|\zeta|}{1+r} \right] \\ &\lesssim b \frac{b^2}{|\log b|} \left(\int (1+r^2) |\varepsilon_1|^2 + |\zeta|^2 + (1+r^2) |\nabla \zeta|^2 + \frac{|\nabla \xi|^2}{1+r^2} \right)^{\frac{1}{2}} \\ &\lesssim \frac{b^3}{|\log b|} \left[C(M) \|\mathbf{E}_2\|_{X_Q} + \frac{b}{|\log b|} \right], \end{aligned}$$

and using an integration by parts:

$$\begin{aligned}
b \left| \int \nabla \hat{\eta} \cdot Q \nabla \mathcal{M}^{(1)} \mathbf{G} \right| &= b |(Q \Delta \hat{\eta} + \nabla Q \cdot \nabla \hat{\eta}, \mathcal{M} E_2)| \\
&\lesssim b \frac{b^2}{|\log b|} \int \left[\frac{|\Delta \hat{\eta}|}{1+r^4} + \frac{|\nabla \hat{\eta}|}{1+r^5} \right] (1+r^2) \\
&\lesssim b \left[\frac{b^2}{|\log b|} C(M) \|\mathbf{E}_2\|_{X_Q} + \frac{b^3}{|\log b|^2} \right].
\end{aligned}$$

5.7. **H terms** : We recall that:

$$\mathbf{H} = -\mathcal{L}\Xi - \frac{\lambda_s}{\lambda} \Lambda \Xi.$$

and the decomposition :

$$\mathcal{L}\Xi = \mathcal{L}\Xi_{sm} + (b - \hat{b})\Lambda \mathbf{Q} + \mathcal{R}$$

where Ξ_{sm} is defined in (4.54) and we have the estimation

$$\mathcal{R} = (b - \hat{b}) \begin{pmatrix} O\left(\frac{\mathbf{1}_{r \geq B_1}}{1+r^4}\right) \\ O\left(\frac{\mathbf{1}_{r \geq B_1}}{1+r^2}\right) \end{pmatrix}.$$

In the following, we notice

$$\mathbf{H}_1 = \mathbf{H} + (b - \hat{b})\Lambda \mathbf{Q} = -\mathcal{L}\Xi_{sm} - \mathcal{R} - \frac{\lambda_s}{\lambda} \Lambda \Xi. \quad (5.38)$$

Moreover from (4.36) and (4.47) :

$$\left| \frac{\lambda_s}{\lambda} \right| \lesssim b \quad \text{and} \quad |b - \hat{b}| \lesssim \frac{b}{|\log b|}.$$

Using together (4.60) and (4.65) yields

$$\int Q \left| \nabla \mathcal{M}^{(1)}(H^{(1)}, H^{(2)}) \right|^2 + \int \left| \Delta \mathcal{M}^{(2)}(H^{(1)}, H^{(2)}) \right|^2 \lesssim \frac{b^4}{|\log b|^2}. \quad (5.39)$$

The bounds (4.56), (4.57) and (4.58) yields

$$\int \frac{|\nabla H_1^{(2)}|^2}{1+r^2} + \int |H_1^{(1)}|^2 \lesssim \frac{b^4}{|\log b|^2}. \quad (5.40)$$

We are in position to estimate the \mathbf{H} terms in (5.29):

$$\begin{aligned}
\left| \langle \widehat{\mathbf{E}}_2, \mathcal{M} \mathcal{L} \mathbf{H} \rangle \right| &= \left| \langle \mathcal{M} \widehat{\mathbf{E}}_2, \mathcal{L} \mathbf{H} \rangle \right| \\
&\lesssim \left(\int Q \left| \nabla \mathcal{M}^{(1)}(\hat{\varepsilon}_2, \hat{\eta}_2) \right|^2 \int Q \left| \nabla \mathcal{M}^{(1)}(H_1^{(1)}, H_1^{(2)}) \right|^2 \right)^{\frac{1}{2}} \\
&+ \left(\int \left| \Delta \mathcal{M}^{(2)}(\hat{\varepsilon}_2, \hat{\eta}_2) \right|^2 \int \left| \Delta \mathcal{M}^{(2)}(H_1^{(1)}, H_1^{(2)}) \right|^2 \right) \\
&\lesssim \frac{1}{100} \left[\int Q \left| \nabla \mathcal{M}^{(1)}(\hat{\varepsilon}_2, \hat{\eta}_2) \right|^2 + \int \left| \Delta \mathcal{M}^{(2)}(\hat{\varepsilon}_2, \hat{\eta}_2) \right|^2 \right] \\
&+ \int Q \left| \nabla \mathcal{M}^{(1)}(H_1^{(1)}, H_1^{(2)}) \right|^2 + \int \left| \Delta \mathcal{M}^{(2)}(H_1^{(1)}, H_1^{(2)}) \right|^2 \\
&\lesssim \frac{1}{100} \left[\int Q \left| \nabla \mathcal{M}^{(1)}(\hat{\varepsilon}_2, \hat{\eta}_2) \right|^2 + \int \left| \Delta \mathcal{M}^{(2)}(\hat{\varepsilon}_2, \hat{\eta}_2) \right|^2 \right] + \frac{b^4}{|\log b|^2}
\end{aligned}$$

In the same way, using (C.1)

$$\begin{aligned} b \left| \int Q \nabla \mathcal{M}^{(1)}(H_1^{(1)}, H_1^{(2)}) \nabla \hat{\eta} \right| &\lesssim \frac{1}{100} \int Q \left| \nabla \mathcal{M}^{(1)}(\hat{\varepsilon}_2, \hat{\eta}_2) \right|^2 + b^2 \int Q |\nabla \hat{\eta}|^2 \\ &\lesssim \frac{1}{100} \int Q \left| \nabla \mathcal{M}^{(1)}(\hat{\varepsilon}_2, \hat{\eta}_2) \right|^2 + C(M) \frac{b^4}{|\log b|^2} \end{aligned}$$

Finally,

$$\begin{aligned} &\left| b \int \frac{\Lambda Q + 2Q}{Q^2} \hat{\varepsilon} \mathcal{L}^{(1)}(H_1^{(1)}, H_1^{(2)}) \right| = \left| b \int \nabla \left\{ \frac{\Lambda Q + 2Q}{Q^2} \hat{\varepsilon} \right\} Q \nabla \mathcal{M}^{(1)}(H_1^{(1)}, H_1^{(2)}) \right| \\ &\lesssim b \left(\int Q \left| \nabla \mathcal{M}^{(1)}(H_1^{(1)}, H_1^{(2)}) \right|^2 \right)^{\frac{1}{2}} \left(\int |\nabla \hat{\varepsilon}|^2 + \frac{|\hat{\varepsilon}|^2}{1+r^2} \right)^{\frac{1}{2}} \\ &\lesssim \frac{b^3}{|\log b|^2} \left[\frac{b}{|\log b|} + C(M) \|\mathbf{E}_2\|_{X_Q} \right]. \end{aligned}$$

For the two last terms, we must use the decomposition (5.38), in order to use the structure of ΛQ . Without this structure, the terms are critical size for our analysis and thus are unmanageable. We shall highlight an additional algebra in order to use the dissipation, to control this pathological term.

First, we are focusing on the \mathbf{H}_1 term, using the estimation (5.40).

$$\begin{aligned} &\left| b \int \hat{\varepsilon}_1 \nabla H_1^{(2)} \right| \\ &\lesssim b^2 \left(\int \frac{|\nabla H_1^{(2)}|^2}{1+\tau^2} \right)^{\frac{1}{2}} \left[\int (1+\tau^2) |\varepsilon_1|^2 + \int (1+\tau^2) |\nabla \zeta|^2 + \int |\zeta|^2 + \int \frac{|\nabla \xi|^2}{1+\tau^2} \right]^{\frac{1}{2}} \\ &\lesssim \frac{b^3}{|\log b|^2} \left[\frac{b}{|\log b|} + C(M) \|\mathbf{E}_2\|_{X_Q} \right]. \end{aligned}$$

Moreover

$$\left| b \int \frac{\Lambda Q + 2Q}{Q^2} \hat{\varepsilon}_2 H_1^{(1)} \right| \lesssim b \|\hat{\varepsilon}_2\|_{L_Q^2} \|H_1^{(1)}\|_{L^2} \lesssim \frac{b^3}{|\log b|^2} \left[\frac{b}{|\log b|} + C(M) \|\mathbf{E}_2\|_{X_Q} \right].$$

The last step is to study this both term for $(b - \hat{b})\Lambda Q$. Its crucial to study them together. Hence,

$$b(b - \hat{b}) \left[\int \frac{\Lambda Q + 2Q}{Q^2} \hat{\varepsilon}_2 \Lambda Q - 2 \int \hat{\varepsilon}_1 \cdot \nabla \phi_{\Lambda Q} \right] = b(b - \hat{b}) \int \hat{\varepsilon}_2 \left[\frac{\Lambda Q (\Lambda Q + 2Q)}{Q^2} + 2\phi_{\Lambda Q} \right].$$

Here we have an additional algebra. Indeed:

$$\frac{\Lambda Q (\Lambda Q + 2Q)}{Q^2} + 2\phi_{\Lambda Q} = \left(\frac{\Lambda Q}{Q} \right)^2 + \mathcal{M}^{(1)}(\Lambda Q, \phi_{\Lambda Q}) = \left(\frac{\Lambda Q}{Q} \right)^2 - 4.$$

As $\int \hat{\varepsilon}_2 = 0$,

$$\int \hat{\varepsilon}_2 \left[\frac{\Lambda Q (\Lambda Q + 2Q)}{Q^2} + 2\phi_{\Lambda Q} \right] = \int \left(\frac{\Lambda Q}{Q} \right)^2 \hat{\varepsilon}_2$$

We have still estimate this term in the subsection 5.4, and we have proved

$$\left| \int \left(\frac{\Lambda Q}{Q} \right)^2 \hat{\varepsilon}_2 \right| \leq \frac{1}{100} \int Q \left| \nabla \mathcal{M}^{(1)}(\hat{\varepsilon}_2, \hat{\eta}_2) \right|^2 + C(M) \frac{b^4}{|\log b|^2}.$$

This concludes the estimations of \mathbf{H} terms.

5.8. **Small linear Θ_b terms.** We recall that:

$$\Theta_b(\varepsilon, \eta) = \begin{vmatrix} \nabla \cdot (\varepsilon \nabla \tilde{\gamma}_b + \tilde{\alpha}_b \nabla \eta) \\ 0 \end{vmatrix} = \begin{vmatrix} \nabla \varepsilon \nabla \tilde{\gamma}_b + \varepsilon \Delta \tilde{\gamma}_b + \tilde{\alpha}_b \Delta \eta + \nabla \tilde{\alpha}_b \nabla \eta \\ 0 \end{vmatrix}.$$

Hence

$$\nabla \phi_{\Theta_b^{(1)}} = \varepsilon \nabla \tilde{\gamma}_b + \tilde{\alpha}_b \nabla \eta.$$

As $\Theta_b^{(2)} = 0$, we have :

$$\mathcal{M}\Theta_b = \begin{vmatrix} \frac{\Theta_b^{(1)}}{Q} \\ -\phi_{\Theta_b^{(1)}} \end{vmatrix}$$

In the proof of the lemma 4.9, we have obtain the rough bounds :

$$\begin{aligned} |\nabla \phi_{\Theta_b^{(1)}}| &\lesssim b \left[\frac{1}{1+r} |\varepsilon| + \frac{|\nabla \eta|}{1+r^2} \right], \\ |\Theta_b^{(1)}| &\lesssim b \left[\frac{1+|\log r|}{1+r} |\nabla \varepsilon| + \frac{1+\log r}{1+r^2} |\varepsilon| + \frac{|\Delta \eta|}{1+r^2} + \frac{|\nabla \eta|}{1+r^3} \right], \\ |\nabla \Theta_b^{(1)}| &\lesssim b \left[\frac{1+|\log r|}{1+r} |\nabla^2 \varepsilon| + \frac{1+\log r}{1+r^2} |\nabla \varepsilon| + \frac{1+\log r}{1+r^3} |\varepsilon| + \frac{|\nabla \Delta \eta|}{1+r^2} + \frac{|\Delta \eta|}{1+r^3} + \frac{|\nabla \eta|}{1+r^4} \right]. \end{aligned}$$

Hence, we obtain the bounds:

$$\int Q \left| \nabla \mathcal{M}^{(1)}(\Theta_b^{(1)}, \Theta_b^{(2)}) \right|^2 + \int \left| \Delta \mathcal{M}^{(2)}(\Theta_b^{(1)}, \Theta_b^{(2)}) \right|^2 \lesssim C(M) b^2 \|\mathbf{E}_2\|_{X_Q}^2 \quad (5.41)$$

and

$$\int \frac{|\nabla \Theta_b^{(2)}|^2}{1+r^2} + \int \left| \Theta_b^{(1)} \right|^2 \lesssim C(M) b^2 \|\mathbf{E}_2\|_{X_Q}^2. \quad (5.42)$$

From the last subsection, these bounds are enough small with respect to b to ensure the control of Θ_b terms.

5.9. **Modulation terms.** We recall that :

$$\widehat{\mathbf{Mod}} = \left(b + \frac{\lambda_s}{\lambda} \right) \Lambda \tilde{\mathbf{Q}}_b - \partial_s \tilde{\mathbf{Q}}_b$$

From the lemma 4.5 and 4.9, we have the following bound:

$$\left| \frac{\lambda_s}{\lambda} + b \right| + |\hat{b}_s| \lesssim C(M) \frac{b^2}{|\log b|}. \quad (5.43)$$

We split the modulation terms in three parts in order to use the structure of the elements of the kernel of the linearized operator \mathcal{L} :

$$\widehat{\mathbf{Mod}} = \widehat{\mathbf{Mod}}_1 + \widehat{\mathbf{Mod}}_2 + \widehat{\mathbf{Mod}}_3$$

with

$$\begin{aligned} \widehat{\mathbf{Mod}}_1 &= \left(b + \frac{\lambda_s}{\lambda} \right) \Lambda \tilde{\mathbf{Q}}_b, \\ \widehat{\mathbf{Mod}}_2 &= -\hat{b}_s \mathbf{T}_1, \\ \widehat{\mathbf{Mod}}_3 &= -\hat{b}_s \left(\tilde{\mathbf{T}}_1 - \mathbf{T}_1 + \hat{b} \frac{\partial \tilde{\mathbf{T}}_1}{\partial b} + 2\hat{b} \tilde{\mathbf{T}}_2 + \hat{b}^2 \frac{\partial \tilde{\mathbf{T}}_2}{\partial b} \right). \end{aligned}$$

From the Proposition 3.2, and (5.43) we have :

$$\begin{aligned}
\left| \widehat{Mod}_1^{(1)} \right| &\lesssim C(M) \frac{b^2}{|\log b|} \left[\frac{1}{1 + \tau^4} + b^2(1 + |\log \tau|) \mathbf{1}_{\tau \leq 2B_1} \right], \\
\left| \nabla \widehat{Mod}_1^{(2)} \right| &\lesssim C(M) \frac{b^2}{|\log b|} \left[\frac{1}{1 + \tau^3} + b^2 \tau(1 + |\log \tau|) \mathbf{1}_{\tau \leq 2B_1} \right], \\
\left| \widehat{Mod}_2^{(1)} \right| &\lesssim C(M) \frac{b^2}{|\log b|} \left[\frac{1}{1 + \tau^2} \right], \\
\left| \nabla \widehat{Mod}_2^{(2)} \right| &\lesssim C(M) \frac{b^2}{|\log b|} \left[\frac{\tau}{1 + \tau^2} \right], \\
\left| \widehat{Mod}_3^{(1)} \right| &\lesssim C(M) \frac{b^2}{|\log b|} \frac{\mathbf{1}_{\tau \geq B_1}}{1 + \tau^2}, \\
&+ C(M) \frac{b^3}{|\log b|} \left\{ \tau^2 \mathbf{1}_{\tau \leq 1} + \frac{1 + |\log(\tau \sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq \tau \leq 6B_0} + \frac{1}{b^2 \tau^4 |\log b|} \mathbf{1}_{\tau \geq 6B_0} \right\}, \\
\left| \nabla \widehat{Mod}_3^{(2)} \right| &\lesssim C(M) \frac{b^2}{|\log b|} \left[\mathbf{1}_{\tau \geq B_1} \frac{\tau}{1 + \tau^2} + b(1 + |\log \tau|) \mathbf{1}_{\tau \leq 2B_1} \right]
\end{aligned}$$

With the above estimations, it is easy to prove :

$$\left\| \frac{\nabla \widehat{Mod}_2^{(2)}}{1 + \tau^2} \right\|_{L^2}^2 + \left\| \mathcal{L} \widehat{Mod}_2 \right\|_{X_Q}^2 + \left\| \tau^i \partial_\tau^i \widehat{Mod}_2^{(1)} \right\|_{L^2}^2 \lesssim C(M) \frac{b^4}{|\log b|^2}, \quad (5.44)$$

$$\left\| \frac{\nabla \widehat{Mod}_3^{(2)}}{1 + \tau^2} \right\|_{L^2}^2 + \left\| \mathcal{L} \widehat{Mod}_3 \right\|_{X_Q}^2 + \left\| \tau^i \partial_\tau^i \widehat{Mod}_3^{(1)} \right\|_{L^2}^2 \lesssim C(M) \frac{b^5}{|\log b|^2}, \quad (5.45)$$

$$\left\| \frac{\nabla \widehat{Mod}_1^{(2)}}{1 + \tau^2} \right\|_{L^2}^2 + \left\| \tau^i \partial_\tau^i \widehat{Mod}_1^{(1)} \right\|_{L^2}^2 \lesssim C(M) \frac{b^4}{|\log b|^2}, \quad (5.46)$$

$$\left\| \mathcal{L} \widehat{Mod}_1 \right\|_{X_Q}^2 \lesssim C(M) \frac{b^5}{|\log b|^2}. \quad (5.47)$$

Remark that we have used for the last bound the cancellation $\mathcal{L} \Lambda \mathbf{Q} = 0$.

Now, we can verify that all modulation terms are manageable in the modified energy (5.29). From (4.62), (5.45), (5.47):

$$\begin{aligned}
\left| \langle \widehat{\mathbf{E}}_2, \mathcal{M} \widehat{\mathcal{L} \mathbf{Mod}} \rangle \right| &\lesssim \left| \langle \mathcal{M} \widehat{\mathbf{E}}_2, \mathcal{L} \widehat{\mathbf{Mod}} \rangle \right| + \|\mathbf{E}_2\|_{X_Q} \left[\left\| \mathcal{L} \widehat{\mathbf{Mod}}_2 \right\|_{X_Q} + \left\| \mathcal{L} \widehat{\mathbf{Mod}}_3 \right\|_{X_Q} \right] \\
&\lesssim \left(\|\mathbf{E}_2\|_{X_Q} + \frac{b^{\frac{3}{2}}}{|\log b|} \right) \left[\left\| \mathcal{L} \widehat{\mathbf{Mod}}_2 \right\|_{X_Q} + \left\| \mathcal{L} \widehat{\mathbf{Mod}}_3 \right\|_{X_Q} \right] \\
&\lesssim C(M) b \left[\frac{b^{\frac{3}{2}}}{|\log b|} \|\mathbf{E}_2\|_{X_Q} + \frac{b^3}{|\log b|^2} \right].
\end{aligned}$$

Moreover, from the bounds (4.56), (4.57) and the interpolation bound (C.1) :

$$\begin{aligned} & b \left| \int \frac{\Lambda Q + 2Q}{Q^2} \left[\hat{\varepsilon} \mathcal{L}^{(1)} \left(\widehat{Mod}^{(1)}, \widehat{Mod}^{(2)} \right) + \hat{\varepsilon}_2 \widehat{Mod}^{(1)} \right] \right| \\ & \lesssim b \left[\|\varepsilon\|_{L^2} \left\| \widehat{\mathcal{L}Mod} \right\|_{X_Q} + \left\{ \|\mathbf{E}_2\|_{X_Q} + \|\mathbf{\Xi}_2\|_{X_Q} \right\} \left\| \widehat{\mathbf{Mod}} \right\|_{L^2} \right] \\ & \lesssim C(M) b \left[\frac{b^{\frac{3}{2}}}{|\log b|} \|\mathbf{E}_2\|_{X_Q} + \frac{b^3}{|\log b|^2} \right]. \end{aligned}$$

Similarly,

$$\begin{aligned} b \left| \hat{\varepsilon}_1 \nabla \widehat{Mod}^{(2)} \right| & \lesssim b \left\| \frac{\nabla \widehat{Mod}^{(2)}}{1 + \tau^2} \right\|_{L^2} \left[\int (1 + \tau^2) |\varepsilon_1|^2 + \int (1 + \tau^2) |\nabla \zeta|^2 + \int |\zeta|^2 + \int \frac{|\nabla \xi|^2}{1 + \tau^2} \right] \\ & \lesssim C(M) \frac{b^3}{|\log b|} \left[\|\mathbf{E}_2\|_{X_Q} + \frac{b}{|\log b|} \right]. \end{aligned}$$

Finally, using the bootstrap bound (4.13) and (4.16)

$$\begin{aligned} & b \left| \int \nabla \hat{\eta} \cdot Q \nabla \mathcal{M}^{(1)} \left(\widehat{Mod}^{(1)}, \widehat{Mod}^{(2)} \right) \right| \lesssim b \int |\nabla \hat{\eta}| \left| |\nabla \widehat{Mod}^{(1)}| + \frac{|\widehat{Mod}^{(1)}|}{1 + \tau} + \frac{|\nabla \widehat{Mod}^{(2)}|}{1 + \tau^4} \right| \\ & \lesssim b \left(\int \frac{|\nabla \eta|^2}{1 + \tau^2} + \frac{|\nabla \xi|^2}{1 + \tau^2} \right)^{\frac{1}{2}} \left(\int (1 + \tau^2) |\nabla \widehat{Mod}^{(1)}|^2 + |\widehat{Mod}^{(1)}|^2 + \frac{|\nabla \widehat{Mod}^{(2)}|^2}{1 + \tau^2} \right)^{\frac{1}{2}} \\ & \lesssim C(M) \frac{b^3}{|\log b|} \left(\|\eta\|_{\dot{H}^1} \|\eta_2\|_{\dot{H}^1} + \frac{b^2}{|\log b|^2} \right)^{\frac{1}{2}} \\ & \lesssim C(M) \frac{b^3}{|\log b|} \left(K^* b^{\frac{5}{2}} |\log b|^2 + \frac{b^2}{|\log b|^2} \right)^{\frac{1}{2}} \\ & \lesssim C(M) \frac{b^4}{|\log b|^2} \end{aligned}$$

5.10. Non-linear \mathbf{N} terms. We recall that

$$\mathbf{N}(\varepsilon, \eta) = \begin{vmatrix} \nabla \cdot (\varepsilon \nabla \eta) \\ 0 \end{vmatrix} \quad \text{and} \quad \nabla \phi_{N^{(1)}}(\varepsilon, \eta) = \varepsilon \nabla \eta.$$

To control the non-linear \mathbf{N} terms, we shall prove the following bounds :

$$\int Q \left| \nabla \mathcal{M}^{(1)}(N^{(1)}, N^{(2)}) \right|^2 + \int \left| \Delta \mathcal{M}^{(2)}(N^{(1)}, N^{(2)}) \right|^2 \lesssim C(M) \frac{b^4}{|\log b|^2} \quad (5.48)$$

and

$$\int \frac{|\nabla N^{(2)}|^2}{1 + \tau^2} + \int \left| N^{(1)} \right|^2 \lesssim C(M) \frac{b^4}{|\log b|^2} \quad (5.49)$$

We have still proved that it is a sufficient condition to be sure that these terms are manageable for our analysis. To prove (5.48) and (5.49), we compute:

$$\begin{cases} \nabla \mathcal{M}^{(1)}(N^{(1)}, N^{(2)}) \\ \nabla \mathcal{M}^{(2)}(N^{(1)}, N^{(2)}) \end{cases} = \begin{cases} \nabla \left(\frac{N^{(1)}(\varepsilon, \eta)}{Q} \right) \\ -\nabla \phi_{N^{(1)}}(\varepsilon, \eta) \end{cases} = \begin{cases} \nabla \left(\frac{\nabla \cdot (\varepsilon \nabla \eta)}{Q} \right) \\ -\varepsilon \nabla \eta \end{cases}$$

Hence

$$\begin{aligned}
& \int Q \left| \nabla \mathcal{M}^{(1)}(N^{(1)}, N^{(2)}) \right|^2 + \int \left| \Delta \mathcal{M}^{(2)}(N^{(1)}, N^{(2)}) \right|^2 \\
&= \int \left| N^{(1)} \right|^2 + \int \frac{[Q \nabla \phi_{N^{(1)}}(\varepsilon, \eta) + \nabla \phi_Q N^{(1)}(\varepsilon, \eta) + \nabla \varepsilon (\Delta \eta + \nabla^2 \eta) + \varepsilon \nabla \Delta \eta]^2}{Q} \\
&\lesssim \int \frac{\varepsilon^2 |\nabla \eta|^2}{1+r^4} + \int (1+r^2) |N^{(1)}(\varepsilon, \eta)|^2 + \int (1+r^4) [\varepsilon^2 |\nabla \Delta \eta|^2 + |\nabla \varepsilon|^2 (|\Delta \eta|^2 + |\nabla^2 \eta|^2)]
\end{aligned}$$

We shall estimate separately each term. First, using (C.7):

$$\begin{aligned}
\left| \int \frac{\varepsilon^2 |\nabla \eta|^2}{1+r^4} \right| &\lesssim \|\nabla \eta\|_{L^\infty}^2 \int \varepsilon^2 \lesssim K^* b^2 |\log b|^6 \|\mathbf{E}_2\|_{X_Q}^2 \\
&\lesssim C(M) \frac{b^4}{|\log b|^2}.
\end{aligned}$$

Now, using the definition of \mathbf{N} , and the bounds (C.7) and (C.6)

$$\begin{aligned}
\left| \int (1+r^2) |N^{(1)}(\varepsilon, \eta)|^2 \right| &\lesssim \int (1+r^2) |\nabla \varepsilon|^2 |\nabla \eta|^2 + \int (1+r^2) |\varepsilon|^2 |\Delta \eta|^2 \\
&\lesssim \|\nabla \eta\|_{L^\infty}^2 \int (1+r^2) |\nabla \varepsilon|^2 + \|(1+r)\varepsilon\|_{L^\infty}^2 \int |\Delta \eta|^2 \\
&\lesssim K^* b^2 |\log b|^6 \|\mathbf{E}_2\|_{X_Q}^2 \\
&\lesssim C(M) \frac{b^4}{|\log b|^2}.
\end{aligned}$$

With the bootstrap bound (4.15) and (C.6), we obtain

$$\begin{aligned}
\left| \int (1+r^4) \varepsilon^2 |\nabla \Delta \eta|^2 \right| &\lesssim \|(1+r)\varepsilon\|_{L^\infty}^2 \int (1+r^2) |\nabla \Delta \eta|^2 \lesssim K^* \frac{b}{\sqrt{|\log b|}} \|\mathbf{E}_2\|_{X_Q}^2 \\
&\lesssim C(M) \frac{b^4}{|\log b|^2}.
\end{aligned}$$

Finally, the bound (C.8) give :

$$\begin{aligned}
\left| \int (1+r^4) |\nabla \varepsilon|^2 (|\Delta \eta|^2 + |\nabla^2 \eta|^2) \right| &\lesssim [\|(1+r)\Delta \eta\|_{L^\infty}^2 + \|(1+r)\nabla^2 \eta\|_{L^\infty}^2] \int (1+r^2) |\nabla \varepsilon|^2 \\
&\lesssim K^* \frac{b}{\sqrt{|\log b|}} \|\mathbf{E}_2\|_{X_Q}^2 \\
&\lesssim C(M) \frac{b^4}{|\log b|^2}.
\end{aligned}$$

Chosen the function δ in (4.6) enough small with respect to K^* , the above estimate concludes the proof of (5.48) and (5.49), and thus the proof of all terms depending on \mathcal{F} .

5.11. **Conclusion.** Injecting all above estimates in (5.29) yields :

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left\{ \frac{\langle \mathcal{M} \widehat{\mathbf{E}}_2, \widehat{\mathbf{E}}_2 \rangle}{\lambda^4} + O\left(\frac{C(M)b^3}{\lambda^4 |\log b|^2}\right) \right\} \\
&\lesssim \frac{bC(M)}{\lambda^6} \left[\frac{b^{\frac{3}{2}}}{|\log b|} \|\mathbf{E}_2\|_{X_Q} + \frac{b^3}{|\log b|^2} \right] + \frac{\hat{b}}{\lambda^6} \int \hat{\varepsilon}_2 \mathcal{M}_\lambda^{(1)}(\hat{\varepsilon}_2, \hat{\eta}_2).
\end{aligned}$$

We are now in position to treat the last term. In this purpose, we shall multiply the above inequality by λ^2 , and we use the following bound coming from (4.36) and (4.47) :

$$\left| \frac{\lambda_s}{\lambda} + \hat{b} \right| \lesssim \frac{b}{|\log b|},$$

in order to obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\{ \frac{\langle \mathcal{M} \widehat{\mathbf{E}}_2, \widehat{\mathbf{E}}_2 \rangle}{\lambda^2} + O\left(\frac{C(M)b^3}{\lambda^4 |\log b|^2}\right) \right\} \lesssim \frac{bC(M)}{\lambda^4} \left[\frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L^2_Q} + \frac{b^3}{|\log b|^2} \right] \\ & + \frac{1}{\lambda^4} \left| \frac{\lambda_s}{\lambda} + \hat{b} \right| \int \hat{\varepsilon}_2 \mathcal{M}_\lambda^{(1)}(\hat{\varepsilon}_2, \hat{\eta}_2) + \frac{1}{\lambda^4} \left| \frac{\lambda_s}{\lambda} \right| \frac{C(M)b^3}{|\log b|^2}. \end{aligned} \quad (5.50)$$

Using the bounds (4.62) and (4.64), we obtain:

$$\begin{aligned} \left| \langle \mathcal{M} \widehat{\mathbf{E}}_2, \widehat{\mathbf{E}}_2 \rangle \right| &= |\langle \mathcal{M} \mathbf{E}_2, \mathbf{E}_2 \rangle| + 2 |\langle \mathcal{M} \Xi_2, \mathbf{E}_2 \rangle| + |\langle \mathcal{M} \Xi_2, \Xi_2 \rangle| \\ &\lesssim |\langle \mathcal{M} \mathbf{E}_2, \mathbf{E}_2 \rangle| + \frac{b^{\frac{3}{2}}}{|\log b|} \|\mathbf{E}_2\|_{X_Q} + \frac{b^3}{|\log b|^2}. \end{aligned} \quad (5.51)$$

In the same way, the bounds (4.61) and (4.63) yield:

$$\begin{aligned} \int \hat{\varepsilon}_2 \mathcal{M}_\lambda^{(1)}(\hat{\varepsilon}_2, \hat{\eta}_2) &= \left| \int \varepsilon_2 \mathcal{M}_\lambda^{(1)}(\varepsilon_2, \eta_2) + 2 \int \varepsilon_2 \mathcal{M}_\lambda^{(1)}(\zeta_2, \xi_2) + \int \zeta_2 \mathcal{M}_\lambda^{(1)}(\zeta_2, \xi_2) \right| \\ &\lesssim \left| \int \varepsilon_2 \mathcal{M}_\lambda^{(1)}(\varepsilon_2, \eta_2) \right| + \frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L^2_Q} + \frac{b^3}{|\log b|^2}. \end{aligned} \quad (5.52)$$

Using the definition of the operator \mathcal{M} and the interpolation bound (C.1), we have:

$$\begin{aligned} \left| \int \varepsilon_2 \mathcal{M}_\lambda^{(1)}(\varepsilon_2, \eta_2) \right| &= \left| \int \frac{|\varepsilon_2|^2}{Q} + \int \varepsilon_2 \eta_2 \right| \\ &\lesssim \int \frac{|\varepsilon_2|^2}{Q} + \left(\int \frac{|\varepsilon_2|^2}{Q} \right)^{\frac{1}{2}} \left(\int Q |\Delta \eta|^2 + \int Q |\varepsilon|^2 \right)^{\frac{1}{2}} \\ &\lesssim \|\mathbf{E}_2\|_{X_Q}^2. \end{aligned} \quad (5.53)$$

Injecting the bounds (5.51), (5.52) and (5.53) in (5.50) yield (5.1). This concludes the proof of the Proposition 5.1

6. Proof of the Proposition 4.4

The bound (4.75) implies $\hat{b}_s < 0$. Using together (4.47) prove the upper bound of (4.75). We prove the non-cancellation of b by contradiction. Suppose that there exist a time $s^* < T$, where T is the maximal time, where the bounds of the bootstrap hold true, such that $b(s^*) = 0$. The bound (4.16) and the interpolation bound (C.1) imply that $\varepsilon(s^*) = 0$. Hence, by conservation of the mass, $\int u(s^*) = \int Q = \int u_0$. This is a contradiction with the initial small super critical mass (4.5). This concludes the proof of (4.75).

6.1. L^1 bound (4.18).

Lemma 6.1 (L^1 bound). *There holds:*

$$\int |\varepsilon| < \frac{1}{2} (\delta^*)^{\frac{1}{4}}. \quad (6.1)$$

Proof. We introduce the decomposition

$$u = \frac{1}{\lambda^2}(Q + \tilde{\varepsilon}) \left(t, \frac{x}{\lambda(t)} \right) \quad \text{i.e. } \tilde{\varepsilon} = \varepsilon + P_{B_1} - Q$$

and we split the perturbation $\tilde{\varepsilon}$ in two parts:

$$\tilde{\varepsilon} = \tilde{\varepsilon}_< + \tilde{\varepsilon}_>, \quad \tilde{\varepsilon}_< = \tilde{\varepsilon} \mathbf{1}_{|\tilde{\varepsilon}| < Q}, \quad \tilde{\varepsilon}_> = \tilde{\varepsilon} \mathbf{1}_{\tilde{\varepsilon} > Q}. \quad (6.2)$$

Using the interpolation bound (C.6), the bootstrap bound (4.11) and the bounds (3.103), (3.107) coming from the construction of the approximate profile, we obtain:

$$\|\tilde{\varepsilon}\|_{L^\infty} \lesssim \|\varepsilon\|_{L^\infty} + |b| \lesssim \delta(\alpha^*). \quad (6.3)$$

Choose a small constant $\eta^* < \delta(\alpha^*)$. Let r such that $|\tilde{\varepsilon}_<(r)| > \eta^* Q(r)$. Then

$$\delta(\alpha^*) \gtrsim \|\tilde{\varepsilon}\|_{L^\infty} > |\tilde{\varepsilon}_<(r)| > \eta^* Q(r).$$

Thus there exist $r(\alpha^*) \rightarrow +\infty$ as $\alpha^* \rightarrow 0$, such that $r(\alpha^*) < r$.

Hence

$$\begin{aligned} \int |\tilde{\varepsilon}_<| &\leq \int |\tilde{\varepsilon}_<| \mathbf{1}_{\eta^* Q < |\tilde{\varepsilon}| < Q} + \int |\tilde{\varepsilon}_<| \mathbf{1}_{|\tilde{\varepsilon}| \leq \eta^* Q} \lesssim \int_{r \geq r(\alpha^*)} Q + \eta^* \int Q \\ &\lesssim \delta(\alpha^*) + \eta^* \lesssim \delta(\alpha^*). \end{aligned}$$

According to the conservation of the mass, the bound (4.5) implies

$$\int \tilde{\varepsilon} < \alpha^*.$$

Now, using that by definition $\tilde{\varepsilon}_> > 0$:

$$\int |\tilde{\varepsilon}| \lesssim \delta(\alpha^*). \quad (6.4)$$

This bound together with the smallness of b (4.11) and the decay of the profiles \tilde{T}_1 and \tilde{T}_2 (3.103), (3.107) imply:

$$\int |\varepsilon| \lesssim \int |\tilde{\varepsilon}| + \sqrt{b} \lesssim \delta(\alpha^*) + \sqrt{\delta^*} < \frac{1}{2}(\delta^*)^{\frac{1}{4}}$$

and concludes the proof of Lemma 6.1. \square

6.2. \dot{H}^1 bound (4.19). We prove this bound with the following monotonicity formula :

$$\frac{d}{dt} \left\{ \frac{1}{\lambda^2} \int |\nabla \eta|^2 + O \left(\frac{b^2 |\log b|^2}{\lambda^2} \right) \right\} \leq \sqrt{K^*} \frac{b^3 |\log b|^6}{\lambda^4} \quad (6.5)$$

Assume (6.5). Notice $\mathcal{E}_1 = \int |\nabla \eta|^2$. We recall the bounds coming from the modulation equations (4.36) and (4.75):

$$|\hat{b}_s| \lesssim \frac{b^2}{|\log b|}, \quad |\lambda \lambda_t + b| \lesssim C(M) \frac{b^2}{|\log b|}. \quad (6.6)$$

Integrating (6.5) in time between 0 and t^* yields:

$$\begin{aligned} \mathcal{E}_1(t) &\leq \frac{\lambda^2(t)}{\lambda^2(0)} [\mathcal{E}_1(0) + O(b(0)^2 |\log b(0)|^2)] \\ &+ \sqrt{K^*} \lambda^2(t) \int_0^{t^*} \frac{b^3 |\log b|^6}{\lambda^4} dt + O(b(t)^2 |\log b(t)|^2). \end{aligned} \quad (6.7)$$

With the bounds (6.6), we estimate the last term :

$$\begin{aligned}
\int_0^{t^*} \frac{b^3 |\log b|^6}{\lambda^4} &= \int_0^{t^*} -\lambda_t \frac{\hat{b}^2 |\log \hat{b}|^6}{\lambda^3} dt + O\left(\int_0^{t^*} \frac{b^4 |\log b|^2}{\lambda^4}\right) \\
&= \left[\frac{\hat{b}^2 |\log \hat{b}|^6}{2\lambda^2}\right]_0^{t^*} - \int_0^{t^*} \frac{1}{2\lambda^4} \frac{d}{ds} \left[\hat{b}^2 |\log \hat{b}|^6\right] dt + O\left(\int_0^{t^*} \frac{b^4 |\log b|^2}{\lambda^4}\right) \\
&= \left[\frac{\hat{b}^2 |\log \hat{b}|^6}{2\lambda^2}\right]_0^{t^*} + O\left(\int_0^{t^*} \frac{b^3 |\log b|^4}{\lambda^4}\right)
\end{aligned}$$

Thus

$$\lambda^2(t) \int_0^{t^*} \frac{b^3 |\log b|^6}{\lambda^4} \lesssim \hat{b}(t)^2 |\log \hat{b}(t)|^6 + \left(\frac{\lambda(t)}{\lambda(0)}\right)^2 \hat{b}(0)^2 |\log \hat{b}(0)|^6 \quad (6.8)$$

Injecting this bound in (6.7), using the smallness of $\mathcal{E}_1(0)$, we obtain :

$$\mathcal{E}_1(t) \leq \sqrt{K^*} \left\{ \hat{b}(t)^2 |\log \hat{b}(t)|^6 + \left(\frac{\lambda(t)}{\lambda(0)}\right)^2 \hat{b}(0)^2 |\log \hat{b}(0)|^6 \right\}.$$

Using the bounds (6.6), we prove without difficulty that :

$$\frac{d}{ds} \left[\frac{\hat{b}^2 |\log \hat{b}|^6}{\lambda^2} \right] > 0. \quad (6.9)$$

This fact coupled with the measure (4.47) of the gap between b and \hat{b} conclude the proof of (4.19).

Proof of (6.5) : We use the second decomposition and as the norm \dot{H}^1 of the flux is invariant by scaling, we have the relation :

$$\int |\nabla \hat{\eta}|^2 = \int |\nabla \hat{z}|^2.$$

Moreover, \hat{z} verifies the equation :

$$\partial_t \hat{z} = \Delta \hat{z} - \hat{w} + \frac{1}{\lambda^2} \left[\Psi_b^{(2)} + \widehat{\text{Mod}}^{(2)} \right]$$

Hence

$$\begin{aligned}
\frac{d}{dt} \frac{1}{2} \int |\nabla \hat{z}|^2 &= - \int \partial_t \hat{z} \Delta \hat{z} \\
&= - \int |\Delta \hat{z}|^2 + \int \Delta \hat{z} \hat{w} - \frac{1}{\lambda^2} \int \Delta \hat{z} \left[\tilde{\Psi}_b^{(2)} + \widehat{\text{Mod}}^{(2)} \right] \\
&= - \int |\Delta \hat{z}|^2 + \int \Delta \hat{z} \hat{w} + \frac{1}{\lambda^2} \int \nabla \hat{\eta} \nabla \left[\tilde{\Psi}_b^{(2)} + \widehat{\text{Mod}}^{(2)} \right]. \quad (6.10)
\end{aligned}$$

Now, using the bootstrap bound (4.16) and the interpolation bound (C.1) :

$$\int \Delta \hat{z} \hat{w} \leq \frac{1}{8} \int |\Delta \hat{z}|^2 + 4 \int \hat{w}^2 \leq \frac{1}{8} \int |\Delta \hat{z}|^2 + \frac{K^*}{\lambda^2} \frac{b^3}{|\log b|^2}. \quad (6.11)$$

Using Cauchy-Schwartz together the bounds (4.13) and (3.114) yields

$$\left| \int \nabla \hat{\eta} \nabla \tilde{\Psi}_b^{(2)} \right| \lesssim \left(\int |\nabla \hat{\eta}|^2 \int |\nabla \tilde{\Psi}_b^{(2)}|^2 \right)^{\frac{1}{2}} \lesssim \sqrt{K^*} b^3 |\log b|^6 \quad (6.12)$$

The bootstrap bound on the modulation parameters (4.38) and the decay of $\nabla\tilde{S}_1$ (3.105) and $\nabla\tilde{S}_2$ (3.109) yields :

$$\begin{aligned} \int |\widehat{\nabla\text{Mod}}^{(2)}|^2 &\lesssim \left| b + \frac{\lambda_s}{\lambda} \right|^2 \int |\nabla\phi_{\Lambda Q} + b\nabla\tilde{S}_1 + b^2\nabla\tilde{S}_2|^2 + |\hat{b}_s|^2 \int |\nabla\tilde{S}_1 + 2b\nabla\tilde{S}_2|^2 \\ &\lesssim \frac{b^4}{|\log b|}. \end{aligned}$$

Hence,

$$\left| \int \nabla\hat{\eta}\widehat{\nabla\text{Mod}}^{(2)} \right| \lesssim \left(\int |\nabla\hat{\eta}|^2 \int |\widehat{\nabla\text{Mod}}^{(2)}|^2 \right)^{\frac{1}{2}} \lesssim \sqrt{K^*} b^3 |\log b|^5 \quad (6.13)$$

Injecting (6.11), (6.12) and (6.13) in (6.10)

$$\frac{d}{dt} \left\{ \int |\nabla\hat{\eta}|^2 \right\} \leq \sqrt{K^*} \frac{b^3 |\log b|^6}{\lambda^2},$$

Now the bound (4.59) yields

$$\frac{d}{dt} \left\{ \int |\nabla\eta|^2 + O(b^2 |\log|^2) \right\} \leq \sqrt{K^*} \frac{b^3 |\log b|^6}{\lambda^2},$$

Dividing this inequality by λ^2 , and using the bound (6.6), we obtain (6.5).

6.3. \dot{H}^2 bound. To prove this bound (4.20), we can use the second decomposition. Hence, we have the equation:

$$\partial_t \hat{z} = \Delta \hat{z} - \hat{w} + \frac{1}{\lambda^2} \left[\tilde{\Psi}_b^{(2)} + \widehat{\text{Mod}}^{(2)} \right]$$

and

$$\partial_t \nabla \hat{z} = \nabla \Delta \hat{z} - \nabla \hat{w} + \frac{1}{\lambda^3} \left[\nabla \tilde{\Psi}_b^{(2)} + \nabla \widehat{\text{Mod}}^{(2)} \right]$$

Next, we compute

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int |\Delta \hat{z}|^2 &= \int \nabla \cdot \{ \nabla \partial_t \hat{z} \} \Delta \hat{z} \\ &= - \int \nabla \partial_t \hat{z} \nabla \Delta \hat{z} \\ &= - \int \left\{ \nabla \Delta \hat{z} - \nabla \hat{w} + \frac{1}{\lambda^3} \left[\nabla \tilde{\Psi}_b^{(2)} + \nabla \widehat{\text{Mod}}^{(2)} \right] \right\} \nabla \Delta \hat{z} \\ &= - \int |\nabla \Delta \hat{z}|^2 + \int \nabla \hat{w} \nabla \Delta \hat{z} + \frac{1}{\lambda^3} \int \left[\nabla \tilde{\Psi}_b^{(2)} + \nabla \widehat{\text{Mod}}^{(2)} \right] \nabla \Delta \hat{z} \end{aligned}$$

Now, using the bootstrap bound (4.16) and the interpolation bound (C.1), we obtain:

$$\left| \int \nabla \hat{\varepsilon} \nabla \Delta \hat{\eta} \right| \leq \frac{1}{10} \int |\nabla \Delta \hat{\eta}|^2 + \int |\nabla \hat{\varepsilon}|^2 + \int |\nabla \zeta|^2 \lesssim K^* \frac{b^3}{|\log b|^2} + \frac{1}{10} \int |\nabla \Delta \hat{\eta}|^2.$$

Now, from the bound of the Proposition 3.2,

$$\int \left| \Delta \tilde{\Psi}_b^{(2)} \right|^2 \lesssim b^5 |\log b|^2$$

and from the bound (4.36) and (4.75)

$$\begin{aligned} \int |\Delta \widehat{Mod}^{(2)}|^2 &\lesssim \left| b + \frac{\lambda_s}{\lambda} \right|^2 \int |\Delta \phi_{\Lambda Q} + b \Delta \tilde{S}_1 + b^2 \Delta \tilde{S}_2|^2 + |\hat{b}_s|^2 \int |\Delta \tilde{S}_1 + 2b \Delta \tilde{S}_2|^2 \\ &\lesssim \frac{b^4}{|\log b|^2}. \end{aligned}$$

Hence

$$\begin{aligned} \left| \frac{1}{\lambda^4} \int \left[\nabla \tilde{\Psi}_b^{(2)} + \nabla \widehat{Mod}^{(2)} \right] \nabla \Delta \hat{\eta} \right| &= \left| \frac{1}{\lambda^4} \int \left[\Delta \tilde{\Psi}_b^{(2)} + \Delta \widehat{Mod}^{(2)} \right] \Delta \hat{\eta} \right| \\ &\lesssim \frac{1}{\lambda^4} \frac{b^2}{|\log b|} \|\Delta \hat{\eta}\|_{L^2} \lesssim \frac{\sqrt{K^*}}{\lambda^4} \frac{b^3}{|\log b|}. \end{aligned}$$

The above estimations together (4.56) and (4.57) give the monotonicity formula:

$$\frac{d}{dt} \left[\frac{1}{\lambda^2} \int |\Delta \eta|^2 + O \left(\frac{b^2}{\lambda^2 |\log b|^2} \right) \right] \lesssim \frac{\sqrt{K^*}}{\lambda^4} \frac{b^3}{|\log b|^2}. \quad (6.14)$$

Using the same proof as the \dot{H}^1 level, *ie* dividing by λ^2 the monotonicity formula and integrating in time, we prove the bound (4.20). The proof is left to the reader.

6.4. \dot{H}^3 bound: We use exactly the same approach than the last subsection. Now, we focus on this equation :

$$\partial_t \Delta \hat{z} = \Delta^2 \hat{z} - \Delta \hat{w} + \frac{1}{\lambda^4} \left[\Delta \tilde{\Psi}_b^{(2)} + \Delta \widehat{Mod}^{(2)} \right].$$

Next, we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int (1+r^2) |\nabla \Delta \hat{z}|^2 = \int (1+r^2) \nabla \partial_t \Delta \hat{z} \nabla \Delta \hat{z} \\ &= - \int \partial_t \Delta \hat{z} \nabla \cdot ((1+r^2) \nabla \Delta \hat{z}) \\ &= - \int \left\{ \Delta^2 \hat{z} - \Delta \hat{w} + \frac{1}{\lambda^4} \left[\Delta \tilde{\Psi}_b^{(2)} + \Delta \widehat{Mod}^{(2)} \right] \right\} \{ 2r \Delta \hat{z} + (1+r^2) \nabla \Delta \hat{z} \} \\ &= - \int (1+r^2) |\Delta^2 \hat{z}|^2 + \int (1+r^2) \Delta^2 \hat{z} \left\{ -\Delta \hat{w} + \frac{1}{\lambda^4} \left[\Delta \tilde{\Psi}_b^{(2)} + \Delta \widehat{Mod}^{(2)} \right] \right\} \\ &\quad - \int 2r \Delta \hat{z} \left\{ \Delta^2 \hat{z} - \Delta \hat{w} + \frac{1}{\lambda^4} \left[\Delta \tilde{\Psi}_b^{(2)} + \Delta \widehat{Mod}^{(2)} \right] \right\} \end{aligned}$$

Using the bound of the last subsection and the bootstrap bound (4.16) :

$$\int (1+r^2) |\Delta \hat{z}|^2 + \int (1+r^2) \left| \Delta \tilde{\Psi}_b^{(2)} \right|^2 + \int (1+r^2) |\Delta \widehat{Mod}^{(2)}|^2 \lesssim K^* \frac{b^3}{|\log b|^2}.$$

In the above estimation, it is very important to see that $\int |\nabla \tilde{S}_1|^2 \sim \log b$, but $\int (1+r^2) |\Delta \tilde{S}_1|^2 \lesssim 1$.

Next, we have, using the bootstrap bound (4.15):

$$\left| \int 2r \Delta \hat{z} \Delta^2 \hat{z} \right| \leq \frac{1}{10} \int (1+r^2) |\Delta^2 \hat{z}|^2 + O \left(\int |\Delta \hat{\eta}|^2 \right) \leq \frac{1}{10} \int (1+r^2) |\Delta^2 \hat{z}|^2 + O \left(K^* \frac{b^2}{|\log b|} \right).$$

Thus, using the bounds (4.56) and (4.57), we obtain:

$$\frac{d}{dt} \left[\frac{1}{\lambda^2} \int (1+r^2) |\nabla \Delta \eta|^2 + O \left(\frac{b^2}{\lambda^4 |\log b|^2} \right) \right] \lesssim \frac{\sqrt{K^*}}{\lambda^4} \frac{b^2}{\sqrt{|\log b|}}. \quad (6.15)$$

Here, the bound is enough large, and we can integrate in time, without dividing by a power of λ , in order to prove (4.21). The proof is left to the reader.

6.5. X_Q **bound** (4.22). The proof of this bound is identical as [36]. We use the same strategy that for the \dot{H}^1 bound. We sketch the argument for the sake of completeness. In the last section, we have proved the following monotonicity formula, where we have use the bootstrap bound (4.16):

$$\frac{d}{dt} \left\{ \frac{(\mathcal{M}\varepsilon_2, \varepsilon_2)}{\lambda^2} + O\left(C(M)\sqrt{K^*} \frac{b^3}{\lambda^2|\log b|^2} \right) \right\} \lesssim \sqrt{K^*} \frac{b^4}{\lambda^4|\log b|^2}. \quad (6.16)$$

First, we integrate in time the last term between 0 and t^* .

$$\begin{aligned} \int_0^{t^*} \frac{b^4}{\lambda^4|\log b|^2} dt &= \int_0^{t^*} -\lambda_t \frac{\hat{b}^3}{\lambda^3|\log \hat{b}|^2} dt + O\left(\int_0^{t^*} \frac{b^5}{\lambda^4|\log b|^5} dt \right) \\ &= \left[\frac{\hat{b}^3}{2\lambda^2|\log \hat{b}|^2} \right]_0^{t^*} - \int_0^{t^*} \frac{1}{2\lambda^4} \frac{d}{ds} \left[\frac{\hat{b}^3}{|\log \hat{b}|^2} \right] dt + O\left(\int_0^{t^*} \frac{b^5}{\lambda^4|\log b|^5} dt \right) \\ &= \left[\frac{\hat{b}^3}{2\lambda^2|\log \hat{b}|^2} \right]_0^{t^*} + O\left(\int_0^{t^*} \frac{b^4}{\lambda^4|\log b|^3} dt \right) \end{aligned}$$

Thus, we obtain:

$$\lambda^2(t) \int_0^{t^*} \frac{b^4}{\lambda^4|\log b|^2} dt \lesssim \frac{b^3(t)}{|\log b(t)|^2} + \left(\frac{\lambda(t)}{\lambda(0)} \right)^2 \frac{b^3(0)}{|\log b(0)|^2}$$

Using the interpolation bound (2.54) and integrating (6.16) in time yield

$$\begin{aligned} \delta(M) \|\varepsilon_2(t)\|_{L^2_Q}^2 &\lesssim (\mathcal{M}\varepsilon_2(t), \varepsilon_2(t)) \quad (6.17) \\ &\lesssim C(M) \left\{ \frac{\lambda^2(t)}{\lambda^2(0)} \left[\|\varepsilon_2(0)\|_{L^2_Q}^2 + \sqrt{K^*} \frac{b^3(0)}{|\log b(0)|^2} \right] + \sqrt{K^*} \frac{b^3(t)}{|\log b(t)|^2} \right\} \end{aligned}$$

for some small enough universal constants $\delta(M), C(M) > 0$ independent of K^* . Moreover, (6.6) implies

$$\frac{d}{ds} \left\{ \frac{\hat{b}^3}{\lambda^2|\log \hat{b}|^2} \right\} > 0 \quad (6.18)$$

and thus (6.17) and the smallness of $\|\mathbf{E}_2(0)\|_{X_Q}$ (4.7) at the initial time yield:

$$\begin{aligned} \delta(M) \|\varepsilon_2(t)\|_{L^2_Q}^2 &\lesssim C(M)\sqrt{K^*} \left[\lambda^2(t) \frac{b^3(0)}{\lambda^2(0)|\log b(0)|^2} + \frac{b^3(t)}{|\log b(t)|^2} \right] \\ &\lesssim C(M)\sqrt{K^*} \left[\lambda^2(t) \frac{\hat{b}^3(0)}{\lambda^2(0)|\log \hat{b}(0)|^2} + \frac{b^3(t)}{|\log b(t)|^2} \right] \\ &\lesssim C(M)\sqrt{K^*} \frac{\hat{b}^3(t)}{|\log \hat{b}(t)|^2} \lesssim C(M)\sqrt{K^*} \frac{b^3(t)}{|\log b(t)|^2} \end{aligned}$$

and (4.22) follows for $K^* = K^*(M)$ large enough. This concludes the proof of Proposition 4.4.

7. Proof of the Theorem 1.1

We are now in position to conclude the proof of Theorem 1.1. The proof follows similar lines as in [36], we sketch the argument for the sake of completeness.

Proof. Step 1 Proof of the blow-up at a finite time T_0 .

Let $u_0 \in \mathcal{O}$ and $u \in \mathcal{C}([0, T], \mathcal{E})$ be the corresponding solution to (1.2) with lifetime $0 < T \leq +\infty$, then the estimates of Proposition 4.4 hold on $[0, T)$. Observe from (6.18) the bound

$$\lambda^2 \lesssim b^3$$

from which using (6.6)

$$-\lambda\lambda_t \gtrsim b \gtrsim C(u_0)\lambda^{\frac{2}{3}} \quad (7.1)$$

and thus

$$-(\lambda^{\frac{4}{3}})_t \gtrsim C(u_0) > 0$$

implies that $\lambda(t)$ touches zero in some finite time $0 < T_0 < +\infty$.

Moreover, as b is a non increasing function, the bounds of Proposition 4.4, the Hardy bound (A.2), the construction of the approximate solution, whose the estimates are available in the Proposition 3.2, and the unique decomposition (4.9) ensure that

$$\|\varepsilon(t)\|_{H^2} + \|\Delta\eta(t)\|_{L^2} \ll 1 \quad \text{for } 0 \leq t < T_0.$$

Furthermore,

$$\lim_{t \uparrow T_0} \|u(t)\|_{H^2} + \|\Delta\eta(t)\|_{L^2} = +\infty$$

and thus using a standard argument from Cauchy theory, the solution blows up at $T = T_0 < +\infty$. Moreover, the bound (6.6) yields:

$$|\lambda\lambda_t| = \left| \frac{\lambda_s}{\lambda} \right| \lesssim 1 \quad \text{and thus } \lambda(t) \lesssim \sqrt{T-t},$$

and thus from (4.23):

$$s(t) \rightarrow +\infty \quad \text{as } t \rightarrow T_0. \quad (7.2)$$

Step 2 Computation of the rate of the concentration.

In the Lemma 4.5 and 4.9, we have obtained equation of modulation parameters (b, \hat{b}, λ) , depending on a suitable norm for the error term. In the bootstrap, we have obtained a enough good bound for this error (4.22) in order to be in position now to reintegrate this equation as $s \rightarrow \infty$, ie in the vicinity of the blow-up time.

$$\left| \hat{b}_s + \frac{2\hat{b}^2}{|\log \hat{b}|} \right| \lesssim \frac{\hat{b}^2}{|\log \hat{b}|^2}. \quad (7.3)$$

This equation is the same as in [?]. Thus, we can use the same strategy to obtain the blow-up speed. First we multiply (7.3) by $\frac{|\log \hat{b}|}{\hat{b}^2}$ and obtain:

$$\frac{\hat{b}_s \log \hat{b}}{\hat{b}^2} = -2 + O\left(\frac{1}{|\log \hat{b}|}\right).$$

We use

$$\left(\frac{\log t}{t} + \frac{1}{t}\right)' = -\frac{\log t}{t^2}$$

to conclude after integration:

$$-\frac{\log \hat{b} + 1}{\hat{b}} = 2s + O\left(\int_0^s \frac{d\sigma}{|\log \hat{b}|}\right).$$

We obtain the equations

$$\frac{-\log \hat{b}}{\hat{b}} = \frac{s}{2} \left(1 + O\left(\frac{1}{\log s}\right)\right) \quad \text{and} \quad \hat{b} = -\frac{2\log \hat{b}}{s} \left(1 + O\left(\frac{1}{\log s}\right)\right),$$

which implies

$$\hat{b}(s) = \frac{\log s}{2s} (1 + o(1)), \quad \log \hat{b} = \log \log s - \log s + O(1).$$

Finally, combining both above estimates, we obtain the development as $s \rightarrow \infty$:

$$\hat{b}(s) = \frac{1}{2s} (\log s - \log \log s) + O\left(\frac{1}{s}\right).$$

Using the bound of the gap between b and \hat{b} (4.47), together the bound on the law λ (4.36) yield

$$-\frac{\lambda_s}{\lambda} = \hat{b} + O\left(\frac{b}{|\log b|}\right) = \frac{1}{2s} (\log s - \log \log s) + O\left(\frac{1}{s}\right). \quad (7.4)$$

Integrating in time this estimation, we obtain

$$-\log \lambda = \frac{1}{4} [(\log s)^2 - 2\log s \log \log s] + O(\log s) = \frac{(\log s)^2}{4} \left[1 - \frac{2\log \log s}{\log s} + O\left(\frac{1}{\log s}\right)\right].$$

Hence,

$$\sqrt{|\log \lambda|} = \frac{\log s}{2} \left[1 - \frac{\log \log s}{\log s} + O\left(\frac{1}{\log s}\right)\right]$$

and thus:

$$e^{2\sqrt{|\log \lambda|} + O(1)} = \frac{s}{\log s}, \quad s = \sqrt{|\log \lambda|} e^{2\sqrt{|\log \lambda|} + O(1)}.$$

We use these relations to rewrite the modulation equation (7.4):

$$-s \frac{\lambda_s}{\lambda} = -\sqrt{|\log \lambda|} e^{2\sqrt{|\log \lambda|} + O(1)} \lambda \lambda_t = \sqrt{|\log \lambda|} + O(1)$$

and thus

$$-\lambda \lambda_t e^{2\sqrt{|\log \lambda|}} = e^{O(1)}. \quad (7.5)$$

The time integration with boundary condition $\lambda(T_0) = 0$ yields

$$\lambda(t) = \sqrt{T_0 - t} e^{-\sqrt{\frac{|\log(T_0 - t)|}{2}} + O(1)} \quad \text{as } t \rightarrow T_0,$$

this is (1.13).

Step 3 Strong convergence (1.12)

Injecting the bound (7.5) in (7.1) implies

$$b(t) \rightarrow 0 \quad \text{as } t \rightarrow T_0.$$

From the estimation of the Proposition 3.2, the above convergence implies

$$\|\mathbf{Q} - \tilde{\mathbf{Q}}_b\|_{W_Q} = \|\tilde{\mathbf{Y}}_b\|_{W_Q} \rightarrow 0 \quad \text{as } t \rightarrow T_0.$$

and the strong convergence (1.12) now follows from the Proposition 4.4.

This concludes the proof of Theorem 1.1. \square

Appendix A. Hardy bounds

In this section, we prove logarithmic Hardy inequalities for radial functions $u \in H_{rad}^3(\mathbb{R}^2)$. They are, with the explicit knowledge of the repulsive structure of the linearized operator \mathcal{L} , the keystone of the proof of the Proposition 2.5, describing the coercivity of this operator under additional orthogonality conditions. This is standard weighted Hardy inequalities, however their proofs are displayed for the reader's convenience.

Lemma A.1 (Weighted Hardy inequality). *There holds the Hardy bounds:*

$$\forall \alpha > -2, \quad \int r^{\alpha+2} |\partial_r v|^2 \geq \frac{(2+\alpha)^2}{4} \int r^\alpha v^2, \quad (\text{A.1})$$

$\forall R > 2, \forall v \in \dot{H}_{rad}^1(\mathbb{R}^2)$ and $\gamma > 0$, there holds the following controls:

$$\int_{r \leq R} \frac{|v|^2}{r^2(1+|\log r|)^2} r dr \lesssim \int_{1 \leq r \leq 2} |v|^2 + \int_{r \leq R} |\nabla v|^2, \quad (\text{A.2})$$

$$\int_{1 \leq r \leq R} \frac{|v|^2}{r^{\gamma+2}(1+|\log r|)^2} r dr \lesssim \int_{1 \leq r \leq 2} |v|^2 + \int_{1 \leq r \leq R} \frac{|\nabla v|^2}{r^\gamma(1+|\log r|)^2}, \quad (\text{A.3})$$

$$\int \frac{v^2}{r^2(1+r^4)(1+|\log r|)^2} \lesssim \int \frac{|\nabla v|^2}{r^4(1+|\log r|)^2} - \int \frac{|v|^2}{1+r^8} \quad (\text{A.4})$$

$\forall v \in H_{rad}^2(\mathbb{R}^2)$ and $\gamma \in [0, 2[$, there holds the bounds:

$$\int \frac{|\nabla v|^2}{r^4(1+|\log r|)^2} + \int \frac{|\nabla^2 v|^2}{r^2(1+|\log r|)^2} \lesssim \int \frac{|\Delta v|^2}{r^2(1+|\log r|)^2} \quad (\text{A.5})$$

$\forall v \in H_{rad}^3(\mathbb{R}^2)$, there holds the Hardy bounds:

$$\int \frac{|\Delta v|^2}{r^2(1+|\log r|)^2} - \int \frac{|\Delta v|^2}{1+r^4} \lesssim \int |\nabla(\Delta v)|^2, \quad (\text{A.6})$$

Proof. (A.1) is a simple consequence of the following integration by parts, for $v \in \mathcal{C}_c^\infty(\mathbb{R}^2)$:

$$\frac{\alpha+2}{2} \int r^\alpha v^2 = - \int r^{\alpha+1} v \partial_r v \leq \left(\int r^\alpha v^2 \right)^{\frac{1}{2}} \left(\int r^{\alpha+2} (\partial_r v)^2 \right)^{\frac{1}{2}}$$

Now, for $v \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, let's prove (A.2). For this purpose, let the radial function $f(r) = -\frac{1}{r(1+|\log(r)|)}$ so that

$$\nabla \cdot f = \begin{cases} \frac{1}{r^2(1+|\log r|)^2} & \text{for } r \geq 1 \\ \frac{-1}{r^2(1+|\log r|)^2} & \text{for } r \leq 1 \end{cases},$$

and integrate by parts to get, with $\varepsilon > 0$:

$$\begin{aligned} & \int_{\varepsilon \leq r \leq R} \frac{|v|^2}{r^2(1+|\log r|)^2} r dr = - \int_{\varepsilon \leq r \leq 1} |v|^2 \nabla \cdot f r dr + \int_{1 \leq r \leq R} |v|^2 \nabla \cdot f r dr \\ &= - \left[\frac{|v|^2}{1+|\log(r)|} \right]_1^R + \left[\frac{|v|^2}{1+|\log(r)|} \right]_\varepsilon^1 + 2 \int_{r \leq R} v \partial_r v \frac{1}{r(1+|\log r|)} r dr \\ &\lesssim |v(1)|^2 + \left(\int_{r \leq R} \frac{|v|^2}{r^2(1+|\log r|)^2} r dr \right)^{\frac{1}{2}} \left(\int_{r \leq R} |\nabla v|^2 r dr \right)^{\frac{1}{2}}. \end{aligned} \quad (\text{A.7})$$

On other hand,

$$|v(1)|^2 \leq \|v\|_{L_{1 \leq r \leq 2}^\infty}^2 \lesssim \int_{1 \leq r \leq 2} |v|^2 + \int_{1 \leq r \leq 2} |\nabla v|^2$$

Injecting this into (A.7) and letting $\varepsilon \rightarrow 0$ yields (A.2). To prove (A.3), let $f(r) = -\frac{e_r}{(1+|\log(r)|)r^{\gamma+1}}$ so that

$$\nabla \cdot f = \frac{2}{(1+|\log r|)^{5\gamma+2}} + \frac{\gamma}{(1+|\log r|)^{2\gamma+2}},$$

and integrate by parts to get:

$$\begin{aligned} & \gamma \int_{1 \leq r \leq R} \frac{|v|^2}{r^{\gamma+2}(1+|\log r|)^2} \leq \int_{1 \leq r \leq R} |v|^2 \nabla \cdot f \\ = & - \left[\frac{|v|^2}{r^\gamma(1+|\log r|)^2} \right]_1^R + 2 \int_{1 \leq r \leq R} v \partial_r v \frac{1}{r^{\gamma+1}(1+|\log r|)^2} \\ \lesssim & |v(1)|^2 + \left(\int_{1 \leq r \leq R} \frac{|v|^2}{r^{\gamma+2}(1+|\log r|)^2} \right)^{\frac{1}{2}} \left(\int_{1 \leq r \leq R} \frac{|\nabla v|^2}{r^\gamma(1+|\log r|)^2} \right)^{\frac{1}{2}}. \end{aligned}$$

We concludes the proof of (A.3) using the same way as the last. The bound (A.4) is a simple consequence of the two last bounds. To prove (A.5), we compute :

$$\int \frac{|\Delta v|^2}{y^2(1+|\log y|)^2} = \int \frac{|\nabla v|^2}{y^4(1+|\log y|)^2} + \int \frac{|\nabla^2 v|^2}{y^2(1+|\log y|)^2} + 2 \int \frac{\partial_r v \partial_{rr} v}{y^3(1+|\log y|)^2}.$$

Now,

$$\begin{aligned} & 2 \int_\varepsilon^R \frac{\partial_r v \partial_{rr} v}{r^3(1+|\log r|)^2} = \int_\varepsilon^R \frac{\partial_r((\partial_r v)^2)}{r^3(1+|\log r|)^2} \\ = & \left[\frac{(\partial_r v)^2}{r^2(1+|\log r|)^2} \right]_\varepsilon^R - \int_\varepsilon^R |\partial_r v|^2 \partial_r \left(\frac{1}{r^2(1+|\log r|)^2} \right) dr \end{aligned}$$

For $u \in H_{rad}^2(\mathbb{R}^2)$, all integrals in the above equality are absolutely convergent. Moreover there exists a sequence $R_n \xrightarrow{n \rightarrow +\infty} +\infty$ such that

$$\left[\frac{(\partial_r v)^2}{r^2(1+|\log r|)^2} \right]_\varepsilon^{R_n} \xrightarrow{n \rightarrow +\infty} 0 - \frac{|\partial_r v(\varepsilon)|^2}{\varepsilon^2(1+|\log \varepsilon|)^2} < 0.$$

Remarking that $\partial_r \left(\frac{1}{r^2(1+|\log r|)^2} \right) < 0$, this implies that, $\forall \varepsilon > 0$

$$2 \int_\varepsilon^R \frac{\partial_r v \partial_{rr} v}{r^3(1+|\log r|)^2} < 0.$$

This concludes the proof of (A.5). The bound (A.6) comes directly from the bound (A.2). \square

Appendix B. On the Poisson field

In this section, we shall write two technical Lemma on the Poisson field. The proof of the first lemma is available in [36].

Lemma B.1 (Interpolation estimates). *Let $u \in L_Q^2$, with $\int u = 0$ then*

$$\|\nabla \phi_u\|_{L^2} \lesssim \|u\|_{L_Q^2}. \quad (\text{B.1})$$

Lemma B.2. *Let $v \in H_{rad}^1$ smooth, such that $\int(1 + |\log r|^2)|\nabla v|^2 < \infty$. Then $\phi_{\Delta v} = v$.*

Proof. Let v satisfying the conditions of the lemma. Let $U = \phi_{\Delta v}$. In the framework of the radial functions, there exist a constant α such that $U = v + \alpha$. Using the convolution representation, we have:

$$\phi_{\Delta v}(0) = \int_0^\infty \log(r)\Delta v(r)rdr.$$

Remark that this integral is absolutely convergent. Now, let $0 < \varepsilon < R < +\infty$.

$$\begin{aligned} \int_\varepsilon^R \log(r)\Delta v(r)rdr &= [r\log(r)v'(r)]_\varepsilon^R - \int_\varepsilon^R v'(r)dr \\ &= R\log Rv'(R) - v(R) - \varepsilon\log\varepsilon v'(\varepsilon) + v(\varepsilon). \end{aligned}$$

As v is smooth, we have

$$\lim_{\varepsilon \rightarrow 0} \varepsilon\log\varepsilon v'(\varepsilon) = 0. \quad (\text{B.2})$$

From the hypothesis of the Lemma:

$$\int |v|^2 + \int (1 + |\log r|^2)|\nabla v|^2 < \infty$$

Hence, there exists a sequence $R_n \rightarrow +\infty$ such that

$$\lim_{n \rightarrow +\infty} R_n \log R_n v'(R_n) - v(R_n) = 0. \quad (\text{B.3})$$

This yields together (B.2) : $\phi_{\Delta v}(0) = v(0)$, and $\alpha = 0$. This concludes the proof of the lemma (B.2). \square

Appendix C. Interpolation bounds

In this section, using the bootstrap bounds of the subsection 4.2, we obtain interpolation bounds, which are the keystone of the proof of the monotonicity formulas.

Proposition C.1 (Interpolation bounds). *(i) W_Q bound :*

$$\begin{aligned} &\int (1 + r^4)|\Delta\varepsilon|^2 + \int (1 + r^2)|\nabla\varepsilon|^2 + \int \varepsilon^2 + \int \frac{|\nabla\phi_\varepsilon|^2}{r^2(1 + |\log r|)^2} \\ &+ \int |\nabla\Delta\eta|^2 + \int \frac{|\Delta\eta|^2}{r^2(1 + |\log r|)^2} + \int \frac{|\nabla\eta|^2}{r^4(1 + |\log r|)^2} \lesssim C(M)\|\mathbf{E}_2\|_{W_Q}^2 \end{aligned} \quad (\text{C.1})$$

$$\int (1 + r^2)|\nabla\varepsilon_1|^2 \lesssim C(M)\|\mathbf{E}_2\|_{W_Q}^2 \quad (\text{C.2})$$

(ii) L^2 bound :¹

$$\int (1 + r^2)|\nabla\Delta\eta|^2 \lesssim K^* \frac{b}{\sqrt{|\log b|}} \quad (\text{C.3})$$

$$\int |\Delta\eta|^2 \lesssim K^* \frac{b^2}{|\log b|} \quad (\text{C.4})$$

$$\int |\nabla\eta|^2 \lesssim K^* b^2 |\log b|^6 \quad (\text{C.5})$$

¹We recall that K^* is the constant of the bootstrap.

(iii) L^∞ bound:

$$\|(1+r)\varepsilon\|_{L^\infty}^2 \lesssim C(M)\|\mathbf{E}_2\|_{W_Q}^2 \quad (\text{C.6})$$

$$\|\nabla\eta\|_{L^\infty}^2 \lesssim K^*b^2|\log b|^6 \quad (\text{C.7})$$

$$\|(1+r)\Delta\eta\|_{L^\infty}^2 + \|(1+r)\nabla^2\eta\|_{L^\infty}^2 \lesssim K^*\frac{b}{\sqrt{|\log b|}} \quad (\text{C.8})$$

Proof. The proof of (C.1) is a simple consequence of the Proposition 2.5 and the bootstrap bound (4.16). The bound (C.2) comes from the following inequality and the bound (C.1) :

$$\int(1+r^2)|\nabla\varepsilon_1|^2 \lesssim \int(1+r^2)|\nabla\varepsilon|^2 + \int\frac{\varepsilon^2}{1+r^2} + \int\frac{|\nabla\eta|^2}{1+r^6}.$$

The bound (C.3), (C.4) and (C.5) are exactly bounds of the bootstrap. To prove (C.6), we have near the origin the estimate, using Sobolev and the bound (C.1) :

$$\|\varepsilon\|_{L_{0\leq r\leq 1}^\infty}^2 \lesssim \|\varepsilon\|_{L_{0\leq r\leq 1}^2}^2 + \|\nabla\varepsilon\|_{L_{0\leq r\leq 1}^2}^2 \lesssim C(M)\|\mathbf{E}_2\|_{W_Q}^2$$

Now, let $f(r) = (1+r)\varepsilon(r)$ and $a \in]1, 2[$ such that:

$$f(a)^2 \leq \int_1^2 |f|^2$$

Let $y \leq 1$. Then

$$f(y)^2 = f(a)^2 + \int_a^y \partial_r(f^2)dr.$$

Using Cauchy-Schwarz,

$$\left| \int_a^y \partial_r(f^2)dr \right|^2 \lesssim \int_1^y \frac{f^2}{r^2} \int_1^y |\partial_r f|^2.$$

Using the definition of f , we obtain

$$\|(1+r)\varepsilon\|_{L^\infty}^2 \lesssim \int|\varepsilon|^2 + \int(1+r^2)|\nabla\varepsilon|^2 \lesssim C(M)\|\mathbf{E}_2\|_{W_Q}^2$$

This concludes the proof of (C.6). Finally, to prove (C.7) and (C.8), we use the same strategy with the bound (C.3), (C.4) and (C.5). This concludes the proof of the Proposition C.1. \square

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